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Nonlinear Analysis 56 (2004) 1011 – 1046

Nonlinear Analysis

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Multiplicity of periodic solutions of nonlinear wave equations

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Received 31 August 2003; accepted 1 November 2003

Abstract

We prove multiplicity of small amplitude periodic solutions, with fixed frequency ω, of completely resonant wave equations with general nonlinearities. As $\omega \to 1$ the number N_ω of $2\pi/\omega$-periodic solutions $u_1, \dots, u_n, \dots, u_{N_\omega}$ tends to $+\infty$. The minimal period of the nth solution u_n is $2\pi/n\omega$. The proofs are based on the variational Lyapunov–Schmidt reduction (Comm. Math. Phys., to appear) and minimax arguments. [1]

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MSC: 35L05; 37K50; 58E05

Keywords: Nonlinear wave equation; Infinite-dimensional Hamiltonian systems; Periodic solutions; Variational methods; Lyapunov–Schmidt reduction

1. Introduction

In a previous paper [5] we have proved, by means of a variational principle, the existence of one, small amplitude periodic solution with *fixed frequency* of the *completely resonant* nonlinear wave equation

$$\begin{cases} u_{tt} - u_{xx} + f(u) = 0, \\ u(t,0) = u(t,\pi) = 0, \end{cases} \tag{1}$$

where $f(0) = f'(0) = 0$.

[1] Supported by M.U.R.S.T. Variational Methods and Nonlinear Differential Equations.

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doi:10.1016/j.na.2003.11.001

The aim of the present paper is to complete the description of the small amplitude periodic solutions of (1) proving "optimal" multiplicity results, finding the minimal periods of the solutions and showing their regularity. These results have been announced in [5, Section 4].

For proving the existence of small amplitude periodic solutions of completely resonant Hamiltonian PDEs like (1) two main difficulties must be overcome. The first is a "small denominator problem" which can be tackled via KAM techniques, see e.g. [7,12,14,15], or Nash–Moser implicit function theorems, see e.g. [6,8,9]; another approach, using the Lindstedt series method, was recently developed in [11]. In [2,3], a strong non-resonance condition on the frequency, which allows the use of the standard contraction mapping theorem, has been introduced. The second problem is the presence of an *infinite-dimensional* space of periodic solutions, with the common period 2π, of the linearized equation at 0

$$\begin{cases} v_{tt} - v_{xx} = 0, \\ v(t,0) = v(t,\pi) = 0. \end{cases} \tag{2}$$

In fact any solution $v = \sum_{j \geqslant 1} \xi_j \cos(jt + \theta_j)\sin(jx)$ of (2) is 2π-periodic in time.

Problem. What linear solutions of V can be continued to solutions of the nonlinear equation (1)?

A natural variational approach to find the bifurcation points of V for general nonlinearities f, has been introduced in [5]. By the classical Lyapunov–Schmidt reduction one has to solve two equations: the (P) equation where, in general, the small denominator problem appears, and the (Q) equation which is the "bifurcation equation" on the infinite dimensional space V. The (P) equation is solved through an implicit function theorem; the small denominator problem is simplified imposing the same strong irrationality condition on the frequency as in [3]. The (Q) equation is solved noting that it is the Euler–Lagrange equation of a *reduced action functional* Φ_ω with *fixed* frequency, see formula (9). Φ_ω is defined in a small neighborhood of 0 and has the form

$$\Phi_\omega(v) = \frac{\omega^2 - 1}{4} \|v\|^2 - G(v) + \text{h.o.t.}, \tag{3}$$

where $G \not\equiv 0$ is an homogeneous functional of order > 2. At least one, non-trivial critical point v of Φ_ω is obtained in [5] by a local mountain-pass argument. Moreover v tends to 0 as $\omega \to 1$. When $f(u) = au^p + \text{h.o.t.}$ and p is *odd*, $G(v) = a/(p+1) \int v^{p+1}$, but, for p *even*, $\int v^{p+1} \equiv 0$ and some more work is required to find the dominant homogeneous term G. *Existence* of periodic orbits is proved for *any* nonlinearity $f(u) = au^p + \text{h.o.t.}$ We remark that the homogeneity of G ensures the "confinement at 0" of the Palais–Smale sequences at the mountain-pass level. This fact is automatically satisfied in finite dimension, see [10].

Multiplicity of critical points is usually derived exploiting the invariance of the functional under some group actions, e.g. the S^1-action induced by time translations in [10,13,16]. We could do it here as well. However, for functionals like (3) with a homogeneous term G we can provide a much more detailed picture of these critical points.

Critical points of Φ_ω restricted to each subspace $V_n \subset V$ formed by the functions of V which are $2\pi/n$-periodic w.r.t. time, are critical points of Φ_ω on the whole V. We prove that, for each $n \geqslant 1$, $\Phi_{\omega|V_n}$ possesses a mountain-pass critical point v_n. Moreover, exploiting the homogeneity of G, we show that each v_n has *minimal period* w.r.t. time $2\pi/n$ and we get a sequence of geometrically distinct $2\pi/\omega$-periodic solutions u_n of (1) (u_n has minimal period $2\pi/n\omega$), see Theorems 3.1–3.4. This method provides very precise informations on the minimal periods, on the norms and on the energies of the solutions, see Remarks 3.1 and 3.6. As $\omega \to 1$, ω fulfilling a strong non-resonance condition, the number N_ω of small amplitude $2\pi/\omega$-periodic solutions $u_1, \ldots, u_n, \ldots, u_{N_\omega}$ we find, increases to $+\infty$ as $N_\omega \approx C/\sqrt{|\omega - 1|} \to +\infty$.

Concerning this estimate for N_ω we remark that the expected number of small amplitude periodic solutions with frequency $\omega \approx 1$ is in general $O(\sqrt{1/|\omega - 1|})$, see Remark 3.2. Actually, for $f(u) = u^3$, all the small amplitude periodic solutions have been computed at the first order (see [3]) showing that we have missed no critical points, see Remark 3.3.

Finally, we prove that the weak solutions found are indeed classical C^2 solutions of (1).

We underline that the forementioned multiplicity results, when $f(u) = au^p + \text{h.o.t.}$ for p *even*, are quite more difficult that the case p odd. The main difficulty is to prove that G satisfies the estimates yielding the minimality of the periods.

The paper is organized as follows. In Section 2, we first recall the variational Lyapunov–Schmidt reduction. We need and prove more refined estimates w.r.t. [5]. Next, we show how to prove the minimality of the period and the multiplicity results. Section 3 concerns the application to periodic solutions of the nonlinear wave equation. In Section 4, we prove the regularity of the solutions.

2. The variational principle

In order to get optimal multiplicity results more refined estimates w.r.t. [5] are needed.

2.1. The variational Lyapunov–Schmidt reduction

We look for periodic solutions of (1) with frequency ω. Normalizing the period to 2π, the problem can be rewritten

$$\omega^2 u_{tt} - u_{xx} + f(u) = 0, \quad u(t,0) = u(t,\pi) = 0, \quad u(t + 2\pi, x) = u(t,x). \tag{4}$$

We look for solutions of (4) $u : \Omega \to \mathbf{R}$ where $\Omega := \mathbf{R}/2\pi\mathbf{Z} \times (0,\pi)$, in the Banach space

$$X := \{u \in H^1(\Omega, \mathbf{R}) \cap L^\infty(\Omega, \mathbf{R}) \mid u(t,0) = u(t,\pi) = 0 \text{ a.e.}, \ u(-t,x) = u(t,x) \text{ a.e.}\}$$

endowed, for $\omega \in \left[\frac{1}{2}, \frac{3}{2}\right]$, $\omega \neq 1$, with the (ω-dependent) norm

$$|u|_\omega := |u|_\infty + |\omega - 1|^{1/2}\|u\|_{H^1}.$$

We can restrict to the space X of functions even in time because Eq. (1) is reversible.

Any $u \in X$ can be developed in Fourier series as

$$u(t,x) = \sum_{l \geqslant 0, j \geqslant 1} u_{lj} \cos lt \sin jx$$

and its H^1-norm and scalar product are

$$\|u\|^2 := \|u\|^2_{H^1} := \int_0^{2\pi} \mathrm{d}t \int_0^\pi \mathrm{d}x\, u_t^2 + u_x^2 = \frac{\pi^2}{2} \sum_{l \geqslant 0, j \geqslant 1} u_{lj}^2 (j^2 + l^2),$$

$$(u,w) := \int_\Omega \mathrm{d}t\,\mathrm{d}x u_t w_t + u_x w_x = \frac{\pi^2}{2} \sum_{l \geqslant 0, j \geqslant 1} u_{lj} w_{lj} (l^2 + j^2) \quad \forall u, w \in X.$$

$|u|_{L^2} := (\int_\Omega |u|^2)^{1/2} = (\pi/\sqrt{2})(\sum_{l \geqslant 0, j \geqslant 1} u_{lj}^2)^{1/2}$ is the L^2-norm, associated with scalar product $(u,w)_{L^2} := \int_\Omega uw$.

The space of the solutions of $v_{tt} - v_{xx} = 0$ which belong to $H_0^1(\Omega, \mathbf{R})$ and are even in time is

$$V := \left\{ v(t,x) = \sum_{j \geqslant 1} \xi_j \cos(jt) \sin jx \mid \xi_j \in \mathbf{R},\ \sum_{j \geqslant 1} j^2 \xi_j^2 < +\infty \right\}.$$

V can also be written as

$$V := \{v(t,x) = \eta(t+x) - \eta(t-x) \mid \eta \in H^1(\mathbf{T}),\ \eta \text{ odd}\},$$

where $\mathbf{T} := \mathbf{R}/2\pi\mathbf{Z}$ and $\eta(s) = \sum_{j \geqslant 1} (\xi_j/2) \sin(js)$. Note that $\langle \eta \rangle := (1/2\pi) \int_{\mathbf{T}} \eta(s)\,\mathrm{d}s = 0$.

In order to get multiplicity results, we shall consider also the subspaces of V

$$V_n := \left\{ v \in V \mid v \text{ is } \frac{2\pi}{n} - \text{periodic w.r.t. } t \right\}, \quad n \in \mathbf{N} \setminus \{0\}.$$

Defining, for $v = \eta(x+t) - \eta(x-t) \in V$, $n \in \mathbf{N} \setminus \{0\}$,

$$(\mathscr{L}_n v)(t,x) := \eta(n(t+x)) - \eta(n(t-x)),$$

it is immediately realized that $\mathscr{L}_n : V \to V_n$ is an isomorphism

$$V_n = \mathscr{L}_n V = \{v(t,x) = \eta(n(x+t)) - \eta(n(x-t)) \mid \eta(\cdot) \in H^1(\mathbf{T}),\ \eta(\cdot) \text{ odd}\}.$$

For $v = \sum \xi_j \cos(jt) \sin(jx) \in V$, we have $|v|_\infty \leqslant \sum |\xi_j| \leqslant C\|v\|$ by the Cauchy–Schwarz inequality. Hence, the norms $\|_\omega$ and $\| \|$ are equivalent on V. Moreover, the embedding $(V, \| \cdot \|) \hookrightarrow (V, | \cdot |_\infty)$ is compact.

Critical points of the C^1-Lagrangian action functional $\Psi : X \to \mathbf{R}$ defined by

$$\Psi(u) := \int_0^{2\pi} \mathrm{d}t \int_0^{\pi} \mathrm{d}x \, \frac{\omega^2}{2} u_t^2 - \frac{1}{2} u_x^2 - F(u),$$

where $F(u) = \int_0^u f(s)\,\mathrm{d}s$, are weak solutions of (4).

To find critical points of Ψ we perform a Lyapunov–Schmidt reduction. Write $X = V \oplus W$ where

$$W := \{ w \in X \mid (w, v)_{L^2} = 0 \ \forall v \in V \}$$
$$= \left\{ \sum_{l \geqslant 0, j \geqslant 1} w_{lj} \cos(lt) \sin jx \in X \mid w_{jj} = 0 \ \forall j \geqslant 1 \right\}.$$

The projectors $\Pi_V : X \to V$, $\Pi_W : X \to W$, defined, for $u = \sum u_{lj} \cos(lt)\sin(jx) \in X$, by $\Pi_V(u) := \sum u_{jj} \cos(jt)\sin(jx)$ and $\Pi_W(u) := u - \Pi_V(u) \in X$, are continuous.

Setting $u := v + w$ with $v \in V$ and $w \in W$, (4) is equivalent to the two equations

$$(\mathrm{Q}) \qquad -\omega^2 v_{tt} + v_{xx} = \Pi_V f(v+w),$$

$$(\mathrm{P}) \qquad -\omega^2 w_{tt} + w_{xx} = \Pi_W f(v+w).$$

The (P) equation is solved through the standard contraction mapping principle, assuming $\omega \in \mathscr{W} := \bigcup_{\gamma > 0} W_\gamma$ where W_γ is the set of strongly non-resonant frequencies introduced in [4]

$$W_\gamma := \left\{ \omega \in \mathbf{R} \mid |\omega l - j| \geqslant \frac{\gamma}{l} \ \forall j \neq l \right\}.$$

For $0 < \gamma < 1/3$, the set W_γ is uncountable, has zero measure and accumulates to $\omega = 1$ both from the left and from the right (see [4]). It is also easy to show that $W_\gamma = \emptyset$ for $\gamma \geqslant 1$.

Lemma 2.1. *For $\omega \in W_\gamma$, the operator $L_\omega := -\omega^2 \partial_{tt} + \partial_{xx} : D(L_\omega) \subset W \to W$ has a bounded inverse $L_\omega^{-1} : W \to W$ which satisfies, for a positive constant C_1 independent of γ and ω, $|L_\omega^{-1} \Pi_W u|_\omega \leqslant (C_1/\gamma)|u|_\omega$ for $u \in X$. Let $L^{-1} : W \to W$ be the inverse operator of $-\partial_{tt} + \partial_{xx}$. There exists $C_2 > 0$ such that $\forall r, s \in X$*

$$\left| \int_\Omega r(t,x)(L_\omega^{-1} - L^{-1})(\Pi_W s(t,x))\,\mathrm{d}t\,\mathrm{d}x \right| \leqslant C_2 \frac{|\omega - 1|}{\gamma} |r|_\omega |s|_\omega, \tag{5}$$

$$\left| \int_\Omega r(t,x) L_\omega^{-1}(\Pi_W s(t,x))\,\mathrm{d}t\,\mathrm{d}x \right| \leqslant C_2 \left(1 + \frac{|\omega - 1|}{\gamma} \right) |r|_\omega |s|_\omega, \tag{6}$$

$$|L_\omega^{-1} \Pi_W s|_{L^2} \leqslant C_2 \left(1 + \frac{\sqrt{|\omega - 1|}}{\gamma} \right) |s|_\omega. \tag{7}$$

Proof. The proof is in the appendix. □

Fixed points of the nonlinear operator $\mathscr{G}: W \to W$ defined by $\mathscr{G}(w) := L_\omega^{-1} \Pi_W f(v+w)$ are solutions of the (P) equation. In order to prove that $\mathscr{G}$ is a contraction for v sufficiently small, we need the following lemma, proved in the appendix, on the Nemytski operator induced by f.

Lemma 2.2. *The Nemytski operator $u \to f(u)$ is in $C^1(X,X)$ and its derivative at u is $Df(u)[h] = f'(u)h$. Moreover, if $f(0) = f'(0) = \cdots = f^{(p-1)}(0) = 0$ there is $\rho_0 > 0$ and positive constants C_3, C_4, depending only on f, such that, $\forall u \in X$ with $|u|_\omega < \rho_0$,*

$$|f(u)|_\omega \leqslant C_3 |u|_\infty^{p-1} |u|_\omega \leqslant C_3 |u|_\omega^p \quad \text{and}$$
$$|f'(u)h|_\omega \leqslant C_4 |u|_\infty^{p-2} |u|_\omega |h|_\omega \leqslant C_4 |u|_\omega^{p-1} |h|_\omega. \tag{8}$$

Lemma 2.3. *Let $f(0) = f'(0) = \cdots = f^{(p-1)}(0) = 0$ and assume that $\omega \in W_\gamma$. There exists $\rho > 0$ such that, $\forall v \in \mathscr{D}_\rho := \{v \in V \mid |v|_\omega^{p-1}/\gamma < \rho\}$ there exists a unique $w(v) \in W$ with $|w(v)|_\omega < |v|_\omega$ solving the (P) equation. Moreover, for some positive constant C_5*

(i) $|w(v)|_\omega \leqslant C_5 |v|_\omega^p/\gamma$, $|w(v)|_{L^2} \leqslant C_5(1 + \sqrt{|\omega - 1|/\gamma})|v|_\omega^p$,
(ii) $\forall r \in X$, $|\int_\Omega (w(v) - L_\omega^{-1} \Pi_W f(v))\, r| \leqslant (C_5/\gamma)(1 + (|\omega - 1|/\gamma))|v|_\omega^{2p-1} |r|_\omega$,
(iii) $w(-v)(t,x) = w(v)(t+\pi, \pi - x)$,
(iv) *$v \in V_n \Rightarrow w(v)$ is $2\pi/n$ periodic w.r.t. t,*
(v) *the map $v \to w(v)$ is in $C^1(V,W)$.*

Proof. The proof is in the appendix. □

Once the (P) equation is solved by $w(v) \in W$, the infinite-dimensional (Q) equation is solved looking for critical points of the *reduced Lagrangian action functional* $\Phi_\omega : \mathscr{D}_\rho \to \mathbf{R}$ defined by

$$\Phi_\omega(v) := \Psi(v + w(v))$$
$$= \int_0^{2\pi} \mathrm{d}t \int_0^\pi \mathrm{d}x \frac{\omega^2}{2} (v_t + (w(v))_t)^2 - \frac{1}{2} (v_x + (w(v))_x)^2 - F(v + w(v)). \tag{9}$$

Indeed it is proved in [5] that

$$D\Phi_\omega(v)[h] = \int_\Omega \mathrm{d}t\, \mathrm{d}x\, \omega^2 v_t h_t - v_x h_x - \Pi_V f(v + w(v))h \quad \forall h \in V. \tag{10}$$

Theorem 2.1 (Berti and Bolle [5]). *If $v \in V$ is a critical point of the reduced action functional $\Phi_\omega : \mathscr{D}_\rho \to \mathbf{R}$ then $u = v + w(v)$ is a weak solution of* (4).

Φ_ω can be written as

$$\Phi_\omega(v) = \frac{\varepsilon}{2} \|v\|^2 + \int_\Omega \mathrm{d}t\, \mathrm{d}x\, \frac{1}{2} f(v + w(v))w(v) - F(v + w(v)), \tag{11}$$

where $\varepsilon := (\omega^2 - 1)/2$.

The existence of one mountain-pass critical point of Φ_ω has been proved in [5]. In the next subsection, we prove a refined abstract theorem which will enable us to find multiplicity results, proving detailed estimate on the norms and the energies of the solutions and considering the minimality of the periods.

2.2. Existence of critical points: an abstract result

Let $\Phi : B_r \subset E \to \mathbf{R}$ be a C^1 functional defined on the ball $B_r := \{v \in E \mid \|v\| < r\}$ of a Hilbert space E with scalar product $(\cdot,\cdot)$ and norm $\|\cdot\|$, of the form

$$\Phi(v) = \frac{\mu}{2}\|v\|^2 - G(v) + R(v), \tag{12}$$

where $G \not\equiv 0$ and

(H1) $G \in C^1(E, \mathbf{R})$ is homogeneous of degree $q+1$ with $q > 1$, i.e. $G(\lambda v) = \lambda^{q+1} G(v)\, \forall \lambda \in \mathbf{R}_+$,
(H2) $\nabla G : E \to E$ is compact,
(H3) $R \in C^1(B_r, \mathbf{R})$, $R(0) = 0$ and for any $r' \in (0, r)$, ∇R maps $B_{r'}$ into a compact subset of E.

The following Theorem was proved in [5].

Theorem 2.2 (Berti and Bolle [5]). *Let G satisfy* (H1), (H2) *and suppose that $G(v) > 0$ for some $v \in E$ (resp. $G(v) < 0$). There is $\alpha > 0$ (depending only on G) and $\mu_0 > 0$ (depending on r and G) such that, for all $R \in C^1(B_r, \mathbf{R})$ satisfying* (H3) *and*

$$|(\nabla R(v), v)| \leqslant \alpha \|v\|^{q+1}, \quad \forall v \in B_r$$

for all $\mu \in (0, \mu_0)$ (resp. $\in (-\mu_0, 0)$), Φ has a non-trivial critical point $v \in B_r$, with $\|v\| = O(\mu^{1/(q-1)})$.

We now prove a more specific result, containing in particular a precise localization of the critical point v of Theorem 2.2. First, we need the following definitions and intermediate lemma.

Observe that, by the compactness of $\nabla G : E \to E$, G maps the unit sphere $S := \{v \in E \mid \|v\| = 1\}$ into a bounded subset of $\mathbf{R}$ (indeed $G(v) = \int_0^1 (\nabla G(sv), v)\, \mathrm{d}s$ and $\nabla G(B)$ is bounded in E, where B denotes the closed unit ball). Define

$$m_+(G) := \sup_{v \in S} G(v) = \sup_{v \neq 0} \frac{G(v)}{\|v\|^{q+1}}, \quad m_-(G) := \inf_{v \in S} G(v) = \inf_{v \neq 0} \frac{G(v)}{\|v\|^{q+1}}.$$

If $m_+(G) > 0$ (resp. $m_-(G) < 0$) then G attains maximum (resp. minimum) on S.

Lemma 2.4. *If $m_+(G) > 0$ then $K_0^+ := \{v \in S \mid G(v) = m_+(G)\}$ is non-empty and compact. If $m_-(G) < 0$ then $K_0^- := \{v \in S \mid G(v) = m_-(G)\}$ is non-empty and compact.*

Proof. We give the proof in the first case. Assume that $m := m_+(G) > 0$. Since $\nabla G(B)$ is compact, $\forall \delta > 0$ there is a finite-dimensional subspace F_δ of E

such that

$$\forall v \in B \ \forall h \in F_\delta^\perp \, |(\nabla G(v), h)| \leqslant \delta \|h\|.$$

As a result, calling P_δ the orthogonal projector onto F_δ, we obtain for $v \in B$

$$\begin{aligned} |G(v) - G(P_\delta v)| &\leqslant \int_0^1 |(\nabla G((1-s)P_\delta v + sv), v - P_\delta v)| \, \mathrm{d}s \\ &\leqslant \int_0^1 \delta \|v - P_\delta v\| \, \mathrm{d}s \leqslant \delta \|v - P_\delta v\| \leqslant \delta. \end{aligned}$$

One can derive from this property that $G_{|B}$ is continuous for the weak topology in E. Indeed $G_{|B}$ is the uniform limit, as $\delta \to 0$, of $G \circ P_\delta$ and each function $G \circ P_\delta$ is continuous for the weak topology since P_δ is a linear compact operator. Since B, the unit ball of E, is compact for the weak topology, G attains its maximum on B at some point $v_0 \in B$. We have $G(v_0) = m > 0$, so that $v_0 \neq 0$, and $G(v_0)/\|v_0\|^{q+1} = m/\|v_0\|^{q+1} \geqslant m$. Hence, by the definition of m, $G(v_0)/\|v_0\|^{q+1} = m$ and $v_0 \in S$, i.e. $v_0 \in K_0^+$.

To prove that K_0^+ is compact, consider a sequence (v_k) with $v_k \in K_0^+$ for all k. Since (v_k) is bounded, we may assume that (up to a subsequence) (v_k) converges to some $v \in E$ for the weak topology. For all k, $v_k \in B$ hence, since B is closed for the weak topology, $v \in B$. Moreover, since $G_{|B}$ is continuous for the weak topology, $G(v) = m$. By the same argument as above, we get that $\|v\| = 1$. Hence, $(v_k) \rightharpoonup v$ and $(\|v_k\|) \to \|v\|$ and we can conclude that $(v_k) \to v$ strongly. $\square$

Note that $m_+(G) > 0$ (resp. $m_-(G) < 0$) if and only if $\exists v \in V$ such that $G(v) > 0$ (resp. $G(v) < 0$).

Theorem 2.3. *Let Φ satisfy* (H1)–(H3) *and suppose that $m := m_+(G) > 0$* (*resp. $m := -m_-(G) > 0$*). *Set $\alpha := \sup_{v \in B_r, v \neq 0} |(\nabla R(v), v)|/\|v\|^{q+1}$. There exists a small positive constant C_0, depending only on q, such that, if*

$$\frac{\alpha}{m} \leqslant C_0, \quad \left(\frac{|\mu|}{m}\right)^{1/(q-1)} \leqslant C_0 r \tag{13}$$

and $\mu > 0$ (*resp. $\mu < 0$*), *Φ has a critical point $v \in B_r$ on a critical level*

$$c = \frac{q-1}{2} m \left(\frac{|\mu|}{(q+1)m}\right)^{(q+1)/(q-1)} \left[1 + O\left(\frac{\alpha}{m}\right)\right]. \tag{14}$$

Moreover

$$v = \left(1 + O\left(\frac{\alpha}{m}\right)\right)\left(\frac{|\mu|}{m(q+1)}\right)^{1/(q-1)} y \quad \text{with } y \in S, \tag{15}$$

$$G(y) = m_+(G) + O(\alpha) \quad (\textit{resp. } G(y) = m_-(G) + O(\alpha)), \tag{16}$$

and $\operatorname{dist}(y, K_0^{\pm}) \leqslant h(\alpha)$ *for some function* h, *depending only on* G, *such that* $\lim_{s\to 0} h(s) = 0$.

Proof. It follows in the main lines the proof of Theorem 2.2 of [5]. (1) Define on the whole space E a new functional $\tilde{\Phi}$ which is an extension of $\Phi_{|\mathscr{U}}$ for some neighborhood $\mathscr{U}$ of 0, in such a way that $\tilde{\Phi}$ possesses the mountain-pass geometry, see (20). (2) Derive by the mountain-pass theorem the existence of a "Palais–Smale" sequence for $\tilde{\Phi}$, see (21). (3) Prove that (v_n) converges to some critical point $\tilde{v}$ in an open ball where $\tilde{\Phi}$ and Φ coincide. (4) Localize $\tilde{v}$, proving (15) and (16).

For definiteness, we make the proof if $G(v) > 0$ for some $v \in V$ (hence $m_+(G) > 0$) and we take $\mu > 0$.

Step 1: Let us consider $\bar{v} \in K_0^+$ such that $G(\bar{v}) = m := m_+(G)$. The function $t \to (\mu/2)\|t\bar{v}\|^2 - G(t\bar{v}) = (\mu/2)t^2 - t^{q+1}m$ attains its maximum at

$$r_\mu := \left(\frac{\mu}{(q+1)m}\right)^{1/(q-1)} \tag{17}$$

with maximum value $((\frac{1}{2}) - 1/(q+1))\mu r_\mu^2$.

Note that $4r_\mu < r$, provided that $(\mu/m)^{1/q-1} \leqslant C_0 r$ for some constant C_0 which depends on q only.

Let $\lambda = [0, +\infty) \to \mathbf{R}$ be a smooth cut-off non-increasing function such that

$$\lambda(s) = 1 \text{ if } s \in [0,4] \quad \text{and} \quad \lambda(s) = 0 \text{ if } s \in [16, +\infty). \tag{18}$$

$\forall \mu > 0$ such that $4r_\mu < r$, we define a functional $\tilde{R}_\mu : E \to \mathbf{R}$ by

$$\tilde{R}_\mu(v) := \lambda\left(\frac{\|v\|^2}{r_\mu^2}\right) R(v) \text{ if } v \in B_r \quad \text{and} \quad \tilde{R}_\mu(v) := 0 \text{ if } v \notin B_r.$$

$\tilde{R}_\mu \in C^1(E, \mathbf{R})$, $\tilde{R}_{\mu|B_{2r_\mu}} = R_{|B_{2r_\mu}}$ and, by (18) and the properties of R, there is a constant C depending on λ and q only, such that

$$\forall v \in E \quad |\tilde{R}_\mu(v)| \leqslant C\alpha\|v\|^{q+1} \quad \text{and} \quad |(\nabla\tilde{R}_\mu(v), v)| + |\tilde{R}_\mu(v)| \leqslant C\alpha r_\mu^{q+1}. \tag{19}$$

Then we can define $\tilde{\Phi}$ on the whole E as

$$\tilde{\Phi}(v) := \frac{\mu}{2}\|v\|^2 - G(v) + \tilde{R}_\mu(v).$$

$\tilde{\Phi}(0) = 0$ and $\tilde{\Phi}$ possesses the mountain-pass geometry: $\exists \delta > 0$ and $w \in \mathbf{R}\bar{v}$ with $\|w\| > \delta$, such that

$$\text{(i)} \quad \inf_{v \in \partial B_\delta} \tilde{\Phi}(v) > 0, \quad \text{(ii)} \tilde{\Phi}(w) < 0. \tag{20}$$

(20)(ii) holds for $w = R\bar{v}$ with R large enough since $\lim_{R\to\infty} \tilde{\Phi}(R\bar{v}) = -\infty$. For (20)(i), we use that $G(v) \leqslant m_+(G)\|v\|^{q+1}$. Hence, by (19), any $\delta \in (0, R)$ such that $(\mu/2)\delta^2 - (m_+(G) + C\alpha)\delta^{q+1} > 0$ is suitable.

Step 2: Define the mountain-pass paths

$$\Gamma = \{\gamma \in C([0,1],E) \mid \gamma(0) = 0,\ \gamma(1) = w\}$$

and the mountain-pass level

$$c_\mu = \inf_{\gamma\in\Gamma} \max_{s\in[0,1]} \tilde{\Phi}(\gamma(s)).$$

By (20)(i), $c_\mu > 0$. By the mountain-pass theorem [1] there exists a sequence (v_n) such that

$$\nabla\tilde{\Phi}(v_n) \to 0, \quad \tilde{\Phi}(v_n) \to c_\mu \tag{21}$$

(Palais–Smale sequence).

Step 3: We shall prove that for n large enough, v_n lies in a ball B_h for some $h < 2r_\mu$. For this we need an estimate of the level c_μ. By the definition of c_μ and (19),

$$c_\mu \leqslant \max_{s\in[0,1]} \tilde{\Phi}(sR\bar{v}) \leqslant \max_{t\in[0,R]} \frac{\mu}{2}\|t\bar{v}\|^2 - (m - C\alpha)\|t\bar{v}\|^{q+1}.$$

Computing the maximum in the right-hand side, we find the estimate

$$c_\mu \leqslant \mu\left(\frac{1}{2} - \frac{1}{q+1}\right)\left(\frac{m}{m - C\alpha}\right)^{2/(q-1)} r_\mu^2. \tag{22}$$

We claim that $\limsup_{n\to+\infty}\|v_n\| < 2r_\mu$. In fact, let $v \in \mathbf{R}_+ \cup \{+\infty\}$ be an accumulation point of the sequence $(\|v_n\|)$. Up to a subsequence, $\lim_{n\to\infty}\|v_n\| = v$ and we shall prove that $v < 2r_\mu$.

By the homogeneity of G, $(\nabla\tilde{\Phi}(v), v) = \mu\|v\|^2 - (q+1)G(v) + (\nabla\tilde{R}_\mu(v), v)$ and, by (21),

$$(\nabla\tilde{\Phi}(v_n), v_n) = \mu\|v_n\|^2 - (q+1)G(v_n) + (\nabla\tilde{R}_\mu(v_n), v_n) = \mu_n\|v_n\|$$

with $\lim_{n\to\infty}\mu_n = 0$. This implies

$$\mu\left(\frac{1}{2} - \frac{1}{q+1}\right)\|v_n\|^2 = \tilde{\Phi}(v_n) - \frac{\mu_n}{q+1}\|v_n\| - \tilde{R}_\mu(v_n) + \frac{1}{q+1}(\nabla\tilde{R}_\mu(v_n), v_n).$$

Since $\tilde{\Phi}(v_n) \to c_\mu$ and using (19), we derive that the sequence $(\|v_n\|)$ is bounded and so $v < \infty$. Taking limits as $n \to \infty$, we obtain

$$\begin{aligned}\mu\left(\frac{1}{2} - \frac{1}{q+1}\right)v^2 &= c_\mu + O(\alpha r_\mu^{q+1}) = c_\mu + O\left(\frac{\alpha}{m}\mu r_\mu^2\right)\\ &\leqslant \mu\left(\frac{1}{2} - \frac{1}{q+1}\right) r_\mu^2\left(1 + O\left(\frac{\alpha}{m}\right)\right)\end{aligned} \tag{23}$$

by (22) and the definition (17) of r_μ. By (23), there is $C_0 > 0$ (depending on q only) such that if $\alpha/m \leqslant C_0$ then $v < 2r_\mu$. More precisely we have

$$v \leqslant r_\mu\left(1 + O\left(\frac{\alpha}{m}\right)\right). \tag{24}$$

Thus $v_n \in B_h$ (for n large), for some $h < 2r_\mu$, and since $\Phi \equiv \tilde{\Phi}$ on B_{2r_μ},

$$\nabla\tilde{\Phi}(v_n) = \nabla\Phi(v_n) = \mu v_n - \nabla G(v_n) + \nabla R(v_n) \to 0.$$

Since (v_n) is bounded, by the compactness assumptions (H2) and (H3), (v_n) converges (up to a subsequence) in B_{2r_μ} to some non-trivial critical point $\tilde{v}$ of Φ at the critical level $c_\mu > 0$.

Step 4: Let us write $\tilde{v} = sy$, with $y \in S$, $s = \|\tilde{v}\| > 0$. We know by (24) that

$$s \leqslant r_\mu \left(1 + O\left(\frac{\alpha}{m}\right)\right). \tag{25}$$

Since

$$\mu\|\tilde{v}\|^2 - (\nabla G(\tilde{v}), \tilde{v}) + (\nabla R(\tilde{v}), \tilde{v}) = (\nabla\Phi(\tilde{v}), \tilde{v}) = 0,$$

$\mu s^2 - (q+1)s^{q+1}G(y) \leqslant \alpha s^{q+1}$ (recall the definition of α). Hence $(q+1)G(y) + \alpha > 0$ and (combined with (25))

$$\frac{\mu}{(q+1)G(y) + \alpha} \leqslant s^{q-1} \leqslant r_\mu^{q-1}\left(1 + O\left(\frac{\alpha}{m}\right)\right). \tag{26}$$

By the definition (17) of r_μ, (26) implies, for α/m small, that $G(y) \geqslant m + O(\alpha)$. Hence, since $G(y) \leqslant m$ by the definition of m (recall that $y \in S$),

$$G(y) = m + O(\alpha)$$

and we have proved (16). Hence, by the first inequality of (26),

$$s^{q-1} \geqslant \frac{\mu}{(q+1)m}\left(1 + O\left(\frac{\alpha}{m}\right)\right).$$

Finally, by the definition (17) of r_μ and (25),

$$s = \|\tilde{v}\| = r_\mu\left(1 + O\left(\frac{\alpha}{m}\right)\right) = \left(\frac{|\mu|}{m(q+1)}\right)^{1/(q-1)}\left(1 + O\left(\frac{\alpha}{m}\right)\right) \tag{27}$$

and we have proved (15).

By (23) and (27),

$$c_\mu = \mu\left(\frac{1}{2} - \frac{1}{q+1}\right)s^2 + O\left(\frac{\alpha}{m}\mu r_\mu^2\right) = \mu\left(\frac{1}{2} - \frac{1}{q+1}\right)r_\mu^2\left(1 + O\left(\frac{\alpha}{m}\right)\right),$$

which yields (14). To complete the proof of Theorem 2.3 note that the estimate $G(y) = m_\pm(G) + O(\alpha)$ implies that $\operatorname{dist}(y, K_0^\pm) \leqslant h(\alpha)$ for some function h, depending only on G, such that $\lim_{s\to 0} h(s) = 0$. In fact, arguing as in the proof of Lemma 2.4, it is easy to prove that from any sequence (y_n) such that $y_n \in S$ and $G(y_n) \to m_\pm(G)$, one can extract a subsequence which converges to some point of $K_0^\pm$. $\square$

2.3. Minimal period

When the Hilbert space E is equal to V, endowed with the H^1 norm, we can give a condition which implies that the critical v provided by Theorem 2.3 has minimal period w.r.t. time 2π.

Lemma 2.5. *Let $\Phi : B_r \subset V \to \mathbf{R}$ satisfy all the hypotheses of Theorem 2.3 and assume furthermore that there is $\beta \in (0,1)$ such that $\forall n \geqslant 2$ and $\forall v \in V$ for which $G(\mathscr{L}_n v) > 0$ (resp. $G(\mathscr{L}_n v) < 0$),*

$$G(\mathscr{L}_n v) \leqslant \beta n^{q+1} G(v) \quad (\textit{resp. } G(\mathscr{L}_n v) \geqslant \beta n^{q+1} G(v)). \tag{28}$$

Then, there is a constant $C_6 \leqslant C_0$ depending only on q and β such that, provided $\alpha/m \leqslant C_6$ and $(|\mu|/m)^{1/(q-1)} \leqslant C_0 r$, the critical point v of Theorem 2.3 has minimal period (w.r.t. t) 2π.

Proof. We give the proof when $m := m_+(G) > 0$ (and for $\mu > 0$).

For $z(t,x) = \eta(t+x) - \eta(t-x) \in V$ we have[2]

$$\begin{aligned}
\|z\|^2 &= \int_0^{2\pi}\int_0^{\pi} (\eta'(t+x) - \eta'(t-x))^2 \, dt \, dx \\
&= \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi} (\eta'(s_1) - \eta'(s_2))^2 \, ds_1 \, ds_2 \\
&= \frac{1}{2}\left[2\pi \int_0^{2\pi} \eta'(s_1)^2 \, ds_1 - 2\int_0^{2\pi} \eta'(s_1) \, ds_1 \right. \\
&\qquad \left. \times \int_0^{2\pi} \eta'(s_2) \, ds_2 + 2\pi \int_0^{2\pi} \eta'(s_2)^2 \, ds_2 \right] \\
&= 2\pi \int_0^{2\pi} \eta'(s)^2 \, ds.
\end{aligned}$$

Hence $\|\mathscr{L}_n z\|^2 = 2\pi \int_0^{2\pi} n^2 \eta'(ns)^2 \, ds = n^2 \|z\|^2$, because η is 2π-periodic. As a consequence, $\forall n \geqslant 2$, $\forall z \in V$ such that $G(\mathscr{L}_n z) > 0$

$$\frac{G(\mathscr{L}_n z)}{\|\mathscr{L}_n z\|^{q+1}} = \frac{G(\mathscr{L}_n z)}{n^{q+1}\|z\|^{q+1}} \leqslant \beta \frac{G(z)}{\|z\|^{q+1}} \leqslant \beta m$$

by (28), and so, $\forall n \geqslant 2$, $\sup_{V_n \cap S} G \leqslant \beta m$.

[2] Lemma 3.3. of [5]. Let $m : \mathbf{R}^2 \to \mathbf{R}$ be 2π-periodic w.r.t. both variables. Then

$$\int_0^{2\pi}\int_0^{\pi} m(t+x, t-x) \, dt \, dx = \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi} m(s_1, s_2) \, ds_1 \, ds_2.$$

By Theorem 2.3 we know that $v=(\mu/m(q+1))^{1/q-1}(1+O(\alpha/m))y$ with $y\in S$ and $G(y)=m+O(\alpha)$. Hence, there is a constant C_6 which depends only on q and β (more precisely $C_6=O(1-\beta)$) such that, if $\alpha/m\leqslant C_6$ then $\sup_{V_n\cap S}G<G(y)$, $\forall n\geqslant 2$. Hence, $\forall n\geqslant 2$, $y\notin V_n$ and y,v have minimal period 2π. $\square$

2.4. Multiplicity of critical points

When Φ is invariant under some symmetry group (e.g. Φ is even), multiplicity of non-trivial critical points can be obtained, as in the symmetric version of the mountain-pass theorem [1]. We remark that the reduced action functional Φ_ω is even. [3] Indeed defining the linear operator $\mathscr{I}:X\to X$ by $(\mathscr{I}u)(t,x):=u(t+\pi,\pi-x)$, $\Psi\circ\mathscr{I}=\Psi$, and, by Lemma 2.3(iii) and since $-v=\mathscr{I}v$,

$$\Phi_\omega(-v)=\Psi(-v+w(-v))=\Psi(\mathscr{I}(v+w(v)))=\Psi(v+w(v))=\Phi_\omega(v).$$

However, we shall adopt a different approach: as explained in the Introduction we shall prove multiplicity considering the restrictions of Φ_ω to the subspaces $V_n\subset V$.

Lemma 2.6. *Any critical point of $\Phi_\omega:V_n\cap\mathscr{D}_\rho\to\mathbf{R}$ is a critical point of $\Phi_\omega:V\cap\mathscr{D}_\rho\to\mathbf{R}$.*

Proof. Let $v\in V_n$ be such that $D\Phi_\omega(v)[h]=0$ $\forall h\in V_n$. We want to prove that $D\Phi_\omega(v)[h]=0$ $\forall h\in V_n^\perp\cap V$. By (10) it is sufficient to prove that $\forall h\in V_n^\perp\cap V$

$$\int_0^\pi \mathrm{d}x\int_0^{2\pi}\mathrm{d}t\, f(v(t,x)+w(v)(t,x))h(t,x)=0.$$

This holds true because, by Lemma 2.3(iv), $f(v(t,x)+w(v)(t,x))$ is $2\pi/n$ periodic w.r.t t and $(t,x)\to h(t,x)\in V_n^\perp\cap V$ does not contain any harmonic with a time frequency multiple of n. $\square$

Define $\Phi_{\omega,n}:\{v\in V\mid \mathscr{L}_n v\in\mathscr{D}_\rho\}\to\mathbf{R}$ by

$$\Phi_{\omega,n}(v):=\Phi_\omega(\mathscr{L}_n v) \tag{29}$$

and introduce the norm

$$|v|_{\omega,n}:=|\mathscr{L}_n v|_\omega=|v|_\infty+|\omega-1|^{1/2}n\|v\|. \tag{30}$$

If $|\omega-1|n^2<1$, which we shall always assume in the sequel, then $|v|_{\omega,n}\leqslant C\|v\|$ for some universal constant C. As a consequence

$$\exists\ \rho_0>0 \text{ such that } \mathscr{D}:=\left\{v\in V\mid \frac{\|v\|^{p-1}}{\gamma}\leqslant\rho_0\right\}\subset\{v\in V\mid \mathscr{L}_n v\in\mathscr{D}_\rho\}, \tag{31}$$

i.e. $\mathscr{D}$ is included in the domain of definition of $\Phi_{\omega,n}$.

[3] Not restricting to the space X of functions even in time the reduced functional would inherit the natural S^1 invariance symmetry defined by time translations.

To get multiplicity results, we shall look, $\forall n \geqslant 1$, for critical points of

$$\Phi_{\omega,n} : \mathscr{D} \to \mathbf{R}.$$

By Lemma 2.6, if $v \in \mathscr{D}$ is a critical point of $\Phi_{\omega,n}$ with minimal period 2π, then $v_n := \mathscr{L}_n v$ is a critical point of Φ_ω with minimal period $2\pi/n$. Hence, by Theorem 2.1, $u_n = v_n + w(v_n)$ is a solution of Eq. (4) with minimal period w.r.t. time $2\pi/n$. Since the solutions u_n have different minimal periods, they define geometrically distinct periodic orbits.

3. Applications to nonlinear wave equations

Let $f(u) = au^p + \text{h.o.t.}$ for some integer $p \geqslant 2$, $a \neq 0$. Recall that $\varepsilon = (\omega^2 - 1)/2$.

Lemma 3.1. *The reduced action functional* $\Phi_\omega : \mathscr{D}_\rho \to \mathbf{R}$ *defined in* (11) *can be developed as*

$$\Phi_\omega(v) = \frac{\varepsilon}{2} \|v\|^2 - \int_\Omega F(v) - \frac{1}{2} \int_\Omega f(v) L^{-1} \Pi_W f(v) + R_\omega(v),$$

with

$$(\nabla R_\omega(v), v) = O\left(\frac{|\varepsilon|}{\gamma} |v|_\omega^{2p} + \frac{|v|_\omega^{3p-1}}{\gamma} \right).$$

Proof. We have, since $\int_\Omega f(v) L^{-1} \Pi_W (f'(v)v) = \int_\Omega f'(v) v L^{-1} \Pi_W f(v)$,

$$\begin{aligned}
(\nabla R_\omega(v), v) &= \int_\Omega [f(v) + f'(v) L^{-1} \Pi_W f(v) - f(v + w(v))] v \\
&= \int_\Omega [f(v) + f'(v) w(v) - f(v + w(v))] v \\
&\quad + \int_\Omega [(L^{-1} - L_\omega^{-1}) \Pi_W f(v)] f'(v) v \\
&\quad + \int_\Omega [L_\omega^{-1} \Pi_W f(v) - w(v)] f'(v) v \\
&= O(|v|_\omega^{p-1} |w(v)|_{L^2}^2) + O\left(\frac{|\varepsilon|}{\gamma} |v|_\omega^p |v|_\omega^p \right) + O\left(\frac{(|\varepsilon| + \gamma) |v|_\omega^{2p-1}}{\gamma^2} |v|_\omega^p \right) \\
&= O\left(\frac{|\varepsilon|}{\gamma^2} |v|_\omega^{3p-1} + |v|_\omega^{3p-1} \right) + O\left(\frac{|\varepsilon|}{\gamma} |v|_\omega^{2p} \right) + O\left(\frac{\gamma + |\varepsilon|}{\gamma^2} |v|_\omega^{3p-1} \right) \\
&= O\left(\frac{|\varepsilon|}{\gamma} |v|_\omega^{2p} + \frac{|v|_\omega^{3p-1}}{\gamma} \right)
\end{aligned}$$

by Lemmas 2.2, 2.3, formula (5) and because $|v|_\omega^{p-1}/\gamma \leqslant \rho$. $\square$

3.1. Case I: p odd

Let $f(u)=au^p+\text{h.o.t.}$ for an *odd* integer p.

Lemma 3.2. *The reduced action functional $\Phi_\omega:\mathscr{D}\to\mathbf{R}$ defined in (11) has the form (12) with $\mu=\varepsilon=(\omega^2-1)/2$,*

$$G(v):=a\int_\Omega \frac{v^{p+1}}{p+1} \quad \text{and} \quad R(v):=R_\omega(v)-\frac{1}{2}\int_\Omega f(v)L^{-1}\Pi_W f(v) - \int_\Omega\left(F(v)-a\,\frac{v^{p+1}}{p+1}\right).$$

Moreover

$$(\nabla R(v),v)=O\left(|v|_\omega^{p+3}+\frac{|\varepsilon|}{\gamma}\,|v|_\omega^{2p}+\frac{|v|_\omega^{3p-1}}{\gamma}\right). \tag{32}$$

Proof. We use Lemma 3.1. We have $\int f(v)v-av^{p+1}=O(|v|_\omega^{p+3})$ since[4] $\int v^{p+2}\equiv 0$. Moreover, $\int f'(v)vL^{-1}\Pi_W f(v)= O(|f'(v)v|_{L^2}|f(v)|_{L^2})=O(|v|_\omega^{2p})=O(|v|_\omega^{p+3})$ since $p\geqslant 3$. $\square$

Lemma 3.3. *$\Phi_{\omega,n}:\mathscr{D}\to\mathbf{R}$ defined in (29) has the form (12), with $\mu=\varepsilon n^2=n^2(\omega^2-1)/2$,*

$$\Phi_{\omega,n}(v)=\frac{\varepsilon n^2}{2}\,\|v\|^2-G(v)+R_n(v) \quad \forall v\in\mathscr{D},$$

where $R_n(v):=R(\mathscr{L}_n v)$. Moreover

$$\begin{aligned}(\nabla R_n(v),v)&=(\nabla R(\mathscr{L}_n v),\mathscr{L}_n v)=O\left(|v|_{\omega,n}^{p+3}+\frac{|\varepsilon|}{\gamma}\,|v|_{\omega,n}^{2p}+\frac{|v|_{\omega,n}^{3p-1}}{\gamma}\right)\\ &=O\left(\|v\|^{p+3}+\frac{|\varepsilon|}{\gamma}\,\|v\|^{2p}+\frac{\|v\|^{3p-1}}{\gamma}\right).\end{aligned} \tag{33}$$

Proof. It is easy to see that

$$G(\mathscr{L}_n(v))=G(v) \quad \forall v\in V,\ n\in\mathbf{N}. \tag{34}$$

Estimate (33) follows by (32) and (30). $\square$

Theorem 2.3 yields the existence of at least one $2\pi/n$-periodic solution of (4) for all $n\geqslant 1$. Define, for $\omega\in\mathscr{W}:=\bigcup_{\gamma>0}W_\gamma$, $\gamma_\omega:=\max\{\gamma\,|\,\omega\in W_\gamma\}$.

[4] By Lemma 3.4 of [5], $v\in V$ and p even $\Rightarrow v^p\in W$, i.e. $\int_\Omega v^p h=0$ $\forall h\in V$.

Theorem 3.1. *Let $f(u) = au^p + h.o.t.$ $(a \neq 0)$ for an odd integer $p \geqslant 3$. There exists a positive constant $C_7 := C_7(f)$ such that, $\forall \omega \in \mathscr{W}$ and $\forall n \in \mathbf{N} \setminus \{0\}$ satisfying*

$$\frac{|\omega - 1|n^2}{\gamma_\omega} \leqslant C_7 \tag{35}$$

and $\omega > 1$ if $a > 0$ (resp. $\omega < 1$ if $a < 0$), Eq. (1) possesses at least one,[5] *even periodic in time classical C^2 solution u_n with minimal period $2\pi/(n\omega)$.*

Proof. G is homogeneous of degree $p+1$ and for $a > 0$ (resp. $a < 0$) $G(v) > 0$ (resp. $G(v) < 0$) $\forall v \neq 0$. For definiteness, assume $a > 0$ and so $\varepsilon > 0$. ∇G and ∇R satisfy the compactness properties (H2) and (H3) as proved in Lemma 3.2 of [5] and so ∇R_n satisfies (H3) as well.

Let us set

$$r = \frac{1}{C_0} \left(\frac{\varepsilon n^2}{m} \right)^{1/p-1}.$$

For $\varepsilon n^2/\gamma$ small (precisely $\varepsilon n^2/\gamma < \rho_0 m C_0^{p-1}$), $B_r \subset \mathscr{D}$ where the domain $\mathscr{D}$ is defined in (31). Moreover, with this choice of r, the second condition in Theorem 2.3 (13) is automatically satisfied (actually $(\varepsilon n^2/m)^{1/p-1} = C_0 r$). We claim that also the first condition in (13) is satisfied provided $|\varepsilon| n^2/\gamma_\omega$ is small enough, because, by (33)

$$\alpha = O\left(r^2 + \frac{\varepsilon}{\gamma} r^{p-1} + \frac{r^{2p-2}}{\gamma} \right) = O\left((\varepsilon n^2)^{2/(p-1)} + \left(\frac{\varepsilon n^2}{\gamma} \right)^2 \right)$$

(recall that $0 < \gamma \leqslant 1$). Applying Theorem 2.3 we deduce the existence of at least one non-trivial critical point $v \in B_r$ of $\Phi_{\omega,n}$. Finally, by (34) also condition (28) is satisfied and, so, by Lemma 2.5, v has minimal period 2π. $\square$

The regularity of the solutions u_n is proved in Section 4.

Remark 3.1 (Energies and norms of the solutions). Using the estimates of Theorem 2.3, it is possible to prove the following estimates[6] on the norm and the energy $\mathscr{E}_n := \int_0^\pi (u_n)_t^2/2 + (u_n)_x^2/2 + F(u_n)\,\mathrm{d}x$ of the solution u_n:

$$\|u_n\|_{H^1} = n \left(\frac{|\omega - 1|n^2}{m(p+1)} \right)^{1/(p-1)} \left(1 + g_1 \left(\frac{|\omega - 1|n^2}{\gamma_\omega} \right) \right),$$

$$\mathscr{E}_n = \frac{n^2}{2\pi} \left(\frac{|\omega - 1|n^2}{m(p+1)} \right)^{2/(p-1)} \left(1 + g_2 \left(\frac{|\omega - 1|n^2}{\gamma_\omega} \right) \right),$$

where $\lim_{s\to 0} g_i(s) = 0$.

[5] Actually a pair of solutions because Φ_ω is even; if $u(t,x)$ is a solution of (1) then so is $u(t+\pi, \pi - x)$. However, we cannot ensure that they are geometrically distinct.

[6] By the choice of r and the fact that $|\mathscr{L}_n v|_\omega = O(\|v\|) = O(r)$, for $|\omega - 1|n^2 < 1$, the critical points v_n of Φ_ω that we obtain satisfy $|v_n|_\omega = O((|\omega - 1|n^2)^{1/(p-1)})$.

By condition (35) we find $N_\omega \approx C\gamma_\omega/\sqrt{|\omega-1|}$ geometrically different $2\pi/\omega$-periodic solutions.

Remark 3.2 (Optimality of N_ω). We expect the estimate on the number N_ω of solutions, with small $|\cdot|_\omega^{p-1}/\gamma_\omega$, to be optimal. Indeed, assume that $N^2|\omega^2-1| \geqslant \gamma_\omega$ and let $V_N = \{\sum_{j=1}^N \xi_j \cos jt \sin jx;\ \ \xi_j \in \mathbf{R}\}$, $B_\delta = \{v \in V \mid |v|_\omega^{p-1}/\gamma < \delta\}$. For $j = l \geqslant N$, $|\omega^2 l^2 - j^2| = |\omega^2-1|j^2 \geqslant |\omega^2-1|N^2 \geqslant \gamma_\omega$, and we can prove that if δ is small, the restriction of Φ_ω to any fibre $(v+V_N^\perp)\cap B_\delta$ is strictly convex if $\omega>1$ (strictly concave if $\omega<1$), so that, for each $v\in V_N \cap B_\delta$, the restricted functional $V_N^\perp \cap B_\delta \ni h \to \Phi_\omega(v+h)$ has at most one critical point $h(v)$. So the critical points of Φ_ω that belong to B_δ are in one-to-one correspondence with the critical points of the map $\Phi_\omega(v+h(v))$ defined in $V_N \cap B_\delta$. We expect the number of the critical points of this map to be $O(N)$.

Remark 3.3. The solutions u_n obtained in Theorem 3.1 are close to some critical point $v(t,x) := \eta(t+x)+\eta(t-x) \in V$ of the functional $(\varepsilon/2)\|v\|^2 - \int_\Omega v^{p+1}/(p+1)$ and so η is close to some critical point of

$$\eta \to 2\pi\varepsilon \int_0^{2\pi} \mathrm{d}s\eta'^2(s) - \frac{a}{2(p+1)}$$

$$\times \sum_{0\leqslant k\leqslant p+1, k\ even} \binom{p+1}{k} \int_0^{2\pi} \mathrm{d}s\eta^k(s) \int_0^{2\pi} \mathrm{d}s\eta^{p-k+1}(s).$$

Such critical points are 2π periodic, odd solutions of some ordinary differential equations and were explicitly computed in [3] for $p=3$.

3.2. *Case II: p even*

The case $f(u) = au^p + \text{h.o.t.}$ with p even is more difficult since $\int_\Omega v^{p+1} \equiv 0$. To find the dominant non-quadratic term in Φ_ω we must distinguish different cases.

Let $f(u) = au^p + \text{h.o.t.}$ for some even p and

(N1) $f^{(d)}(0) = bd! \neq 0$ for some odd integer $d < 2p-1$ and $f^{(r)}(0)=0$ for any odd integer $r<d$ (for instance $f(u) = au^p + bu^d$),

(N2) $f^{(r)}(0) = 0$ for any odd integer $r \leqslant 2p-1$ (for instance $f(u)=au^p$),

(N3) $f^{(r)}(0)=0$ for any odd integer $r<2p-1$ and $f^{(2p-1)}(0) =: b(2p-1)! \neq 0$ (for instance $f(u) = au^p + bu^{2p-1}$).

(a) *Case* (N1): If (N1) holds the dominant term in the reduced functional is supplied by the odd nonlinearity bu^d in the Taylor expansion of f at 0 and one reduces to the situation discussed in the previous subsection.

Theorem 3.2. *Let $f(u) = au^p + h.o.t.$ with p even and* (N1) *hold. There exists a positive constant $C_8 := C_8(f)$ such that, $\forall \omega \in \mathscr{W}$ and $\forall n \in \mathbf{N}\backslash\{0\}$*

satisfying

$$\frac{(|\omega - 1|n^2)^{(p-1)/(d-1)}}{\gamma_\omega} \leqslant C_8 \tag{36}$$

and $\omega > 1$ *if* $b > 0$ (*resp.* $\omega < 1$ *if* $b < 0$), *Eq.* (1) *possesses at least one*, *even periodic in time classical* C^2 *solution* u_n *with minimal period* $2\pi/(n\omega)$.

Proof. By Lemma 3.1 the reduced action functional Φ_ω can be written in the form (12) with $\mu = \varepsilon$, $G(v) := (b/d+1)\int_\Omega v^{d+1}$ (where, by (N1), $b := f^{(d)}(0)/d!$), $q = d$ and, since $\int_\Omega v^{d+2} \equiv 0$ and $d+3 \leqslant 2p$,

$$(\nabla R(v), v) = O\left(|v|_\omega^{d+3} + \frac{\varepsilon}{\gamma} |v|_\omega^{2p} + \frac{|v|_\omega^{3p-1}}{\gamma} \right).$$

If $b > 0$ (resp. $b < 0$) $G(v) > 0$ (resp. $G(v) < 0$) for all $v \neq 0$. Moreover $G(\mathscr{L}_n v) = G(v)$. Hence, $\Phi_{\omega,n}(v) = (\varepsilon n^2/2)\|v\|^2 - G(v) + R_n(v)$ and, as in the proof of (33),

$$(\nabla R_n(v), v) = O\left(\|v\|^{d+3} + \frac{|\varepsilon|}{\gamma} \|v\|^{2p} + \frac{\|v\|^{3p-1}}{\gamma} \right).$$

Arguing as in the proof of Theorem 3.1, with

$$r = \frac{1}{C_0} \left(\frac{|\varepsilon| n^2}{m} \right)^{1/d-1} \quad \text{and} \quad \alpha = O\left(r^2 + \frac{|\varepsilon|}{\gamma} r^{2p-1-d} + \frac{r^{3p-2-d}}{\gamma} \right)$$

for some small enough positive constant C_8, condition (36) ensures that $B_r \subset \mathscr{D}$ and that conditions (13) hold (observe that $2p-1-d \geqslant 2$). Hence, Theorem 2.3 implies the existence of at least one critical point of $\Phi_{\omega,n}$ which, by Lemma 2.5 (recall that $G(\mathscr{L}_n v) = G(v)$), has minimal period 2π. □

When (N1) is not satisfied the situation is more delicate.

(b)
Case (N2):

Lemma 3.4. *Let* $f(u) = au^p + h.o.t.$ *for an even integer* p *and* (N2) *hold. The reduced action functional* $\Phi_\omega : \mathscr{D} \to \mathbf{R}$ *defined in* (11) *has the form* (12) *with* $\mu = \varepsilon = (\omega^2 - 1)/2$,

$$G(v) := \frac{a^2}{2} \int_\Omega v^p L^{-1} v^p,$$

$$R(v) := R_\omega(v) - \int_\Omega F(v) - \frac{1}{2} \int_\Omega f(v) L^{-1} \Pi_W f(v) + \frac{a^2}{2} \int_\Omega v^p L^{-1} v^p.$$

Moreover

$$(\nabla R(v), v) = O\left(|v|_\omega^{2p+2} + \frac{|\varepsilon|}{\gamma} |v|_\omega^{2p} + \frac{|v|_\omega^{3p-1}}{\gamma} \right). \tag{37}$$

Proof. Since p is even (see footnote 4), $\forall v \in V$, $v^p \in W$ and hence $\Pi_W v^p = v^p$. By (N2) $f^{(p+1)}(0) = 0$, hence $f(v) = av^p + O(v^{p+2})$. We obtain

$$\frac{1}{2}\int_\Omega f(v)L^{-1}\Pi_W f(v) = \frac{a^2}{2}\int_\Omega v^p L^{-1} v^p + r(v) \quad \text{with } (\nabla r(v), v) = O(|v|_\omega^{2p+2}).$$

Moreover, $\int_\Omega v^l = 0$ for all odd l and, by (N2), the first even term in the expansion of $F(v)$ is $O(v^{2p+2})$. Hence $\int_\Omega F(v) = O(|v|_\omega^{2p+2})$. We derive the claim from Lemma 3.1. □

$\nabla G, \nabla R : V \to V$ are still compact operators and G is homogeneous of degree $q + 1 := 2p$. The following Lemma, proved in Lemma 3.5 of [5], ensures that $G \not\equiv 0$, implying the existence of at least one critical point v of Φ_ω, see Theorem 3.2 of [5].

Lemma 3.5 (Berti and Bolle [5]). *For any even* p, $G(v) = (a^2/2)\int_\Omega v^p L^{-1} v^p < 0$ $\forall v \neq 0$.

Further difficulties arise for proving that the minimal period of v is 2π, i.e. that G satisfies (28).

Lemma 3.6. *Let* $w(t,x) \in W$ *be of the form* $w(t,x) = m(t+x, t-x)$ *for some* $m : \mathbf{R}^2 \to \mathbf{R}$ 2π*-periodic with respect to both variables. There exists functions* $\tilde m$ *and* a, 2π*-periodic, such that*

$$m(s_1,s_2) = \tilde m(s_1,s_2) + a(s_1) + a(s_2) + \langle m\rangle, \tag{38}$$

where $\langle m\rangle := (2\pi)^{-2}\int_{\mathbf{T}^2} m(s_1,s_2)\,ds_1\,ds_2$, $\langle a\rangle := (2\pi)^{-1}\int_{\mathbf{T}} a(s)\,ds = 0$, $\langle \tilde m\rangle_{s_1}(s_2) = 0$, $\langle \tilde m\rangle_{s_2}(s_1) = 0$. *Moreover*

$$\begin{aligned}
&\int_0^{2\pi}\int_0^{\pi} L^{-1}(w)w\,dt\,dx \\
&\quad = -\frac{1}{2}\int_0^{2\pi}\int_0^{2\pi} M(s_1,s_2)\tilde m(s_1,s_2)\,ds_1\,ds_2 \\
&\qquad + 2\pi\int_0^{2\pi} M(s,s)a(s)\,ds \\
&\qquad + 2\pi\langle m\rangle\int_0^{2\pi} M(s,s)\,ds - 8\pi\int_0^{2\pi} A(s)^2\,ds - \frac{\langle m\rangle^2\pi^4}{6},
\end{aligned} \tag{39}$$

where M *and* A *are the* 2π*-periodic functions defined by*

$$\begin{aligned}
&A'(s) = \tfrac{1}{4}a(s), \quad \langle A\rangle = 0, \quad \partial_{s_1}\partial_{s_2}M(s_1,s_2) = \tfrac{1}{4}\tilde m(s_1,s_2), \\
&\langle M\rangle_{s_1}(s_2) = 0, \quad \langle M\rangle_{s_2}(s_1) = 0.
\end{aligned} \tag{40}$$

Proof. The proof is in the appendix. □

As a consequence of Lemma 3.6 we obtain:

Lemma 3.7. *Let $w(t,x)\in W$ be of the form $w(t,x)=m(t+x,t-x)$ for some $m:\mathbf{R}^2\to\mathbf{R}$ 2π-periodic with respect to both variables. For $n\in\mathbf{N}$ define $(\mathscr{L}_n w)(t,x):=m(n(t+x),n(t-x))$. Then*

$$\int_0^{2\pi}\int_0^{\pi}(\mathscr{L}_n w)L^{-1}(\mathscr{L}_n w)\,\mathrm{d}x\,\mathrm{d}t$$

$$=-\frac{\pi^4}{6}\langle m\rangle^2+\frac{1}{n^2}\left(\int_0^{2\pi}\int_0^{\pi}wL^{-1}w\,\mathrm{d}x\,\mathrm{d}t+\frac{\pi^4}{6}\langle m\rangle^2\right) \tag{41}$$

$$=-\frac{\pi^4}{6}\langle m\rangle^2+O\left(\frac{|w|_{L^2}^2}{n^2}\right). \tag{42}$$

Proof. We have $(\mathscr{L}_n w)(t,x)=m_n(t+x,t-x)$, with $m_n(s_1,s_2):=m(ns_1,ns_2)$. Using the decomposition given in the latter lemma, we can write $m_n(s_1,s_2)=\tilde{m}_n(s_1,s_2)+a_n(s_1)+a_n(s_2)+\langle m\rangle$, where $\tilde{m}_n(s_1,s_2):=m(ns_1,ns_2)$ and $a_n(s):=a(ns)$. Therefore, by (39), using the abbreviation $w_n:=\mathscr{L}_n w$,

$$\int_0^{2\pi}\int_0^{\pi}L^{-1}(w_n)w_n\,\mathrm{d}x\,\mathrm{d}t$$

$$=-\frac{1}{2}\int_0^{2\pi}\int_0^{2\pi}M_n(s_1,s_2)\tilde{m}_n(s_1,s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2$$

$$+2\pi\int_0^{2\pi}M_n(s,s)a_n(s)\,\mathrm{d}s$$

$$+2\pi\langle m\rangle\int_0^{2\pi}M_n(s,s)\,\mathrm{d}s-8\pi\int_0^{2\pi}A_n(s)^2\,\mathrm{d}s-\frac{\langle m\rangle^2\pi^4}{6},$$

where $M_n(s_1,s_2):=M(ns_1,ns_2)/n^2$ and $A_n(s):=A(ns)/n$. Eq. (41) follows straightforwardly. Eq. (42) follows since $\int_\Omega wL^{-1}w=O(|w|_{L^2}|L^{-1}w|_{L^2})=O(|w|_{L^2}^2)$ and $\langle m\rangle^2=O(|m|_{L^2}^2)=O(|w|_{L^2}^2)$. □

Lemma 3.7 allows to prove that when $p=2$, G satisfies condition (28) in Lemma 2.5.

Lemma 3.8. *Let $f(u)=u^2+h.o.t.$ and* (N2) *hold (i.e. $f^{(3)}(0)=0$). Then*

$$G(v)\leqslant G(\mathscr{L}_n v)<0\ \ \forall v\in V\setminus\{0\}\ \ \forall n\geqslant 2. \tag{43}$$

Proof. We have $v^2(t,x)=m(t+x,t-x)\in W$ with $m(s_1,s_2)=(\eta(s_1)-\eta(s_2))^2$. We can then decompose m as in (38) with $\langle m\rangle=2\langle\eta^2\rangle$, $a(s)=\eta^2(s)-\langle\eta^2\rangle$ and $\tilde{m}(s_1,s_2)=-2\eta(s_1)\eta(s_2)$.

Let P_1, resp. P_2, be the primitive of η, resp. $\eta^2 - \langle \eta^2 \rangle$, with zero mean-value, i.e. $\langle P_1 \rangle = \langle P_2 \rangle = 0$. The functions A, M defined in (40) are here $A(s) := P_2(s)/4$ and $M(s_1, s_2) = -P_1(s_1)P_1(s_2)/2$. Applying Lemma 3.6, we can compute[7]

$$\int_\Omega v^2 L^{-1} v^2 + \frac{\pi^4}{6} \langle m \rangle^2 = -\pi \int_0^{2\pi} P_1(s)^2 (\eta(s)^2 + \langle \eta^2 \rangle)\, \mathrm{d}s$$
$$- \frac{\pi}{2} \int_0^{2\pi} P_2(s)^2\, \mathrm{d}s \leqslant 0. \tag{44}$$

By Lemma 3.7 and (44)

$$\int_\Omega (\mathscr{L}_n v)^2 L^{-1} (\mathscr{L}_n v)^2 = -\frac{\pi^4}{6} \langle m \rangle^2 + \frac{1}{n^2} \left(\int_\Omega v^2 L^{-1} v^2 + \frac{\pi^4}{6} \langle m \rangle^2 \right)$$
$$\geqslant -\frac{\pi^4}{6} \langle m \rangle^2 + \left(\int_\Omega v^2 L^{-1} v^2 + \frac{\pi^4}{6} \langle m \rangle^2 \right) = \int_\Omega v^2 L^{-1} v^2$$

and (43) follows. □

We do not know whether (43) still holds true for G with $p \geqslant 4$. In any case, from Lemma 3.5, we get:

Lemma 3.9. *$\forall n \geqslant 2$, $G_n(v) := G(\mathscr{L}_n v) = (a^2/2) \int (\mathscr{L}_n v)^p L^{-1} (\mathscr{L}_n v)^p$ satisfies (28) with $\beta = \frac{1}{4}$, i.e.*

$$\frac{m^{2p}}{4} G_n(v) \leqslant G_n(\mathscr{L}_m v) < 0 \quad \forall n, m \geqslant 2 \ \ \forall v \in V \setminus 0.$$

Proof. By (41)

$$G_n(v) = G(\mathscr{L}_n v) = \frac{G(v)}{n^2} + \frac{a^2 \pi^4 \langle v^p \rangle^2}{12} \left(\frac{1}{n^2} - 1 \right)$$

with $\langle v^p \rangle := (1/2\pi^2) \int_\Omega v^p$. Hence $\forall n, m \geqslant 2$, $\forall v \in V \setminus \{0\}$ (recall that $G(v) \leqslant 0$),

$$G_n(\mathscr{L}_m v) = G(\mathscr{L}_{mn} v) = \frac{G(v)}{(nm)^2} + \frac{a^2 \pi^4 \langle v^p \rangle^2}{12} \left(\frac{1}{(nm)^2} - 1 \right) \geqslant \frac{1}{4} m^{2p} G_n(v).$$

Indeed $G(v)/(nm)^2 \geqslant G(v) m^{2p}/(4n^2)$ and $(1/(nm)^2) - 1 \geqslant -1 \geqslant (m^{2p}/4)((1/n^2) - 1)$ $\forall n, m \geqslant 2$. □

Lemma 3.10. *Let $f(u) = au^p +$ h.o.t. for an even integer p and* (N2) *hold. Then $\Phi_{\omega,n} : \mathscr{D} \to \mathbf{R}$ defined in (29) has the form (12) with $\mu = \varepsilon n^2 = n^2(\omega^2 - 1)/2$:*

$$\Phi_{\omega,n}(v) = \frac{\varepsilon n^2}{2} \|v\|^2 - G_n(v) + R_n(v) \quad \forall v \in \mathscr{D},$$

[7] We check again from (44) that $G(v) < 0$ $\forall v \neq 0$.

where $G_n(v) = G(\mathscr{L}_n v)$ *and* $R_n(v) = R(\mathscr{L}_n v)$. *Moreover*

$$(\nabla R_n(v), v) = (\nabla R(\mathscr{L}_n v), \mathscr{L}_n v) = O\left(\|v\|^{2p+2} + \frac{|\varepsilon|}{\gamma}\|v\|^{2p} + \frac{\|v\|^{3p-1}}{\gamma} \right).$$

Proof. It follows from Lemma 3.4. □

We derive the following theorem.

Theorem 3.3. *Let* $f(u) = au^p + h.o.t$ *with* p *even and* (N2) *hold. There is a constant* C_9 *depending only on* f *such that*, $\forall \omega \in \mathscr{W}$, $\omega < 1$, $\forall n \geqslant 2$ *such that*

$$\frac{(|\omega - 1|n^2)^{1/2}}{\gamma_\omega} \leqslant C_9, \tag{45}$$

Eq. (1) *possesses at least one, even periodic in time classical* C^2 *solution* u_n *with minimal period* $2\pi/(n\omega)$. *If* $p = 2$ *the existence result holds true for* $n = 1$ *as well.*

Proof. Consider frequencies $\omega < 1$ so that $\varepsilon < 0$. For $(|\varepsilon|n^2)^{1/2}/\gamma$ small enough we can apply Theorem 2.3 and Lemma 2.5 to $\Phi_{\omega,n} : B_r \subset \mathscr{D} \to \mathbf{R}$ with $q = 2p - 1$, $\mu = \varepsilon n^2$,

$$r = \frac{1}{C_0}\left(\frac{|\varepsilon|n^2}{|m_-(G_n)|} \right)^{1/(2p-2)} \quad \text{and} \quad \alpha = O\left(r^2 + \frac{|\varepsilon|}{\gamma} + \frac{r^{p-1}}{\gamma} \right).$$

Note that $m_-(G_n) \to m_-(\bar{G})$ as $n \to +\infty$, where $\bar{G}(v) := -(a^2\pi^4/12)\langle v^p \rangle^2 = -(a^2/48)(\int v^p)^2$, and the sequence $(|m(G_n)|)$ is bounded from below by a positive constant. Moreover, by Lemma 3.9, for $n \geqslant 2$, $G_n(v)$ satisfies (28) with $\beta = \frac{1}{4}$ (and by Lemma 3.8, if $p = 2$, G too satisfies (28)). The Theorem is proved. □

Remark 3.4. We think that the restriction $n \geqslant 2$ in the case $p \neq 2$ even is of purely technical nature.

(c) *Case* (N3): Arguing as in Lemma 3.4 we get:

Lemma 3.11. *Let* $f(u) = au^p + h.o.t.$ *for an even integer* p *and* (N3) *hold. The reduced action functional* $\Phi_\omega : \mathscr{D} \to \mathbf{R}$ *defined in* (11) *has the form* (12) *with* $\mu = \varepsilon = (\omega^2 - 1)/2$,

$$G(v) := \frac{b}{2p}\int_\Omega v^{2p} + \frac{a^2}{2}\int_\Omega v^p L^{-1} v^p \tag{46}$$

and $R(v)$ *satisfies estimate* (37).

By Lemma 3.5, for $b < 0$, $G(v) < 0 \forall v \neq 0$. Moreover Lemma 3.9 yields

Lemma 3.12. *Let* $b < 0$. *For* $n \geqslant 2$, $G_n(v) := G(\mathscr{L}_n v)$ *satisfies* (28) *with* $\beta = \frac{1}{4}$, *i.e.*

$$\tfrac{1}{4} m^{2p} G_n(v) \leqslant G_n(\mathscr{L}_m v) < 0 \quad \forall n, m \geqslant 2, \ \forall v \neq 0.$$

For $\omega < 1$, i.e. $\varepsilon < 0$, and $(|\varepsilon|n^2)^{1/2}/\gamma$ small enough we get the same existence result as in Theorem 3.3.

For $b > 0$ we are no longer able to prove that condition (28) of Lemma 2.5 is satisfied by G_n for any $n \geqslant 2$. That is why we introduce another homogeneous map $\tilde{G}$, which is a good approximation of G_n for large n.

Lemma 3.13. *Let $f(u) = au^p + h.o.t.$ for an even integer p and* (N3) *hold. $\Phi_{\omega,n} : \mathscr{D} \to \mathbf{R}$ defined in* (29) *has the form* (12) *with $\mu = \varepsilon n^2 = n^2(\omega^2 - 1)/2$:*

$$\Phi_{\omega,n}(v) = \frac{\varepsilon n^2}{2} \|v\|^2 - \tilde{G}(v) + R_n(v),$$

where, for $v(t,x) = \eta(t+x) - \eta(t-x) \in V$,

$$\tilde{G}(v) = \frac{b}{2p} \int_\Omega v^{2p} - \frac{a^2}{48} \left(\int_\Omega v^p \right)^2$$
$$= \frac{b}{4p} \int_{\mathbf{T}^2} (\eta(s_1) - \eta(s_2))^{2p} - \frac{a^2}{192} \left(\int_{\mathbf{T}^2} (\eta(s_1) - \eta(s_2))^p \right)^2.$$

Moreover

$$(\nabla R_n(v), v) = (\nabla R(\mathscr{L}_n v), \mathscr{L}_n v) + O\left(\frac{|v|^{2p}_{\omega,n}}{n^2} \right)$$
$$= O\left(\|v\|^{2p+2} + \frac{|\varepsilon|}{\gamma} \|v\|^{2p} + \frac{\|v\|^{3p-1}}{\gamma} + \frac{\|v\|^{2p}}{n^2} \right).$$

Proof. By (42) with $m(s_1, s_2) = (\eta(s_1) - \eta(s_2))^p$

$$\int_\Omega (\mathscr{L}_n v)^p L^{-1} (\mathscr{L}_n v)^p = -\frac{\pi^4}{6} \langle m \rangle^2 + O\left(\frac{|v^p|^2_{L^2}}{n^2} \right)$$
$$= -\frac{1}{96} \left(\int_{\mathbf{T}^2} (\eta(s_1) - \eta(s_2))^p \right)^2 + O\left(\frac{\|v\|^{2p}}{n^2} \right). \tag{47}$$

By (46) and (47) we find the expression of $\tilde{G}$ and the estimate for R_n. $\square$

Clearly, $\tilde{G}(\mathscr{L}_n v) = \tilde{G}(v)$ $\forall v \in V$ and $\forall n \in \mathbf{N} \setminus \{0\}$. We have to specify the sign of $\tilde{G}(v)$.

Lemma 3.14. *It results, for p even,*

$$\kappa(p) := \sup_{v \in V \setminus \{0\}} \left(\int_\Omega v^p \right)^2 \left(\int_\Omega v^{2p} \right)^{-1}$$

$$= \sup_{\eta \in H^1(\mathbf{T}), \eta \ odd} \left(\int_{\mathbf{T}^2} (\eta(s_1) - \eta(s_2))^p \right)^2 \left(2 \int_{\mathbf{T}^2} (\eta(s_1) - \eta(s_2))^{2p} \right)^{-1}$$

$$= \pi^2.$$

$$\inf_{v \in V \setminus \{0\}} \left(\int_\Omega v^p \right)^2 \left(\int_\Omega v^{2p} \right)^{-1} = 0. \tag{48}$$

Proof. Using that η is odd, we get by Newton's formula

$$\int_{\mathbf{T}\times\mathbf{T}} (\eta(s_1) - \eta(s_2))^p \, \mathrm{d}s_1 \, \mathrm{d}s_2$$

$$= \sum_{k=0}^{p/2} C_p^{2k} \int_{\mathbf{T}} \eta^{2k} \, \mathrm{d}s \int_{\mathbf{T}} \eta^{p-2k} \, \mathrm{d}s \leqslant \sum_{k=0}^{p/2} C_p^{2k} 2\pi \int_{\mathbf{T}} \eta^p \, \mathrm{d}s$$

by Hölder inequality. Hence, since $\sum_{k=0}^{p/2} C_p^{2k} = 2^{p-1}$,

$$\int_{\mathbf{T}\times\mathbf{T}} (\eta(s_1) - \eta(s_2))^p \, \mathrm{d}s_1 \, \mathrm{d}s_2 \leqslant 2^p \pi \int_{\mathbf{T}} \eta^p \, \mathrm{d}s. \tag{49}$$

Now, by Cauchy–Schwarz inequality, for all $0 \leqslant k \leqslant p$,

$$\left(\int_{\mathbf{T}} \eta^p \right)^2 = \left(\int_{\mathbf{T}} |\eta|^k |\eta|^{p-k} \right)^2 \leqslant \int_{\mathbf{T}} \eta^{2k} \int_{\mathbf{T}} \eta^{2p-2k}.$$

Hence, since $\sum_{k=0}^{p} C_{2p}^{2k} = 2^{2p-1}$,

$$\int_{\mathbf{T}\times\mathbf{T}} (\eta(s_1) - \eta(s_2))^{2p} \, \mathrm{d}s_1 \, \mathrm{d}s_2$$

$$= \sum_{k=0}^{p} C_{2p}^{2k} \int_{\mathbf{T}} \eta^{2k} \, \mathrm{d}s \int_{\mathbf{T}} \eta^{2p-2k} \, \mathrm{d}s \geqslant 2^{2p-1} \left(\int_{\mathbf{T}} \eta^p \right)^2. \tag{50}$$

By (49) and (50) we obtain $\kappa(p) \leqslant \pi^2$. Choosing for η H^1-approximations of the 2π-periodic function that takes value -1 on $(-\pi, 0]$ and value 1 on $(0, \pi]$, we obtain the converse inequality. Hence $\kappa(p) = \pi^2$.

We now prove (48). $\forall \delta > 0$ there is an odd smooth 2π-periodic η_δ such that $\int_0^{2\pi} \eta_\delta^{2p} = 1$ and $\int_0^{2\pi} \eta_\delta^{2q} \leqslant \delta$ for $1 \leqslant q < p$. One can see easily that for $v(t,x) = \eta_\delta(t+x) - \eta_\delta(t-x)$, $(\int_\Omega v^p)^2 (\int_\Omega v^{2p})^{-1} \leqslant \beta(\delta)$ with $\lim_{\delta \to 0} \beta(\delta) = 0$. $\square$

If $b \geqslant p\kappa(p)a^2/24 = p\pi^2 a^2/24$ then $\tilde{G}(v) > 0$ $\forall v \in V \setminus 0$, and we obtain an existence result for $\omega > 1$.

If $0 < b < p\kappa(p)a^2/24 = p\pi^2 a^2/24$ then there is $v_1 \in V$ such that $\tilde{G}(v_1) < 0$ and there is $v_2 \in V$ such that $\tilde{G}(v_2) > 0$. In this case, we obtain an existence result both for $\omega < 1$ and $\omega > 1$.

In both cases, we apply Theorem 2.3 for $(|\varepsilon|n^2)^{1/2}/\gamma$ small (in order to have $B_r \subset \mathscr{D}$) and n large enough, with $q = 2p-1$, $\mu = \varepsilon n^2$,

$$r = \frac{1}{C_0}\left(\frac{|\varepsilon|n^2}{|m_\pm(\tilde{G})|}\right)^{1/(2p-2)} \quad \text{and} \quad \alpha = O\left(r^2 + \frac{\varepsilon}{\gamma} + \frac{r^{p-1}}{\gamma} + \frac{1}{n^2}\right).$$

Recalling that (28) holds for $\tilde{G}$, we obtain

Theorem 3.4. *Let $f(u) = au^p +$ h.o.t. with p even and* (N3) *hold. Define $J \subset \mathbf{R}$ as $J=(0,1)$ if $b<0$, $J=(1,2)$ if $b \geqslant p\pi^2a^2/24$ and $J=(0,1)\cup(1,2)$ if $0<b<p\pi^2a^2/24$. There is a constant C_{10} and an integer $N_0 \geqslant 1$ depending only on f (with $N_0 = 2$ if $b<0$ and $p \geqslant 4$, $N_0 = 1$ if $b<0$ and $p=2$) such that, $\forall \omega \in \mathscr{W} \cap J$, $\forall n \geqslant N_0$ such that*

$$\frac{(|\omega-1|n^2)^{1/2}}{\gamma_\omega} \leqslant C_{10}, \tag{51}$$

Eq. (1) *possesses at least one even periodic in time classical C^2 solution u_n with minimal period $2\pi/(n\omega)$.*

Remark 3.5. By the choice of r in the different cases and the fact that $|\mathscr{L}_n v|_\omega = O(\|v\|) = O(r)$, for $|\omega-1|n^2 < 1$, the critical points v_n of Φ_ω that we obtain satisfy $|v_n|_\omega = O((|\omega-1|n^2)^{1/(q-1)})$ with $q = d$ in case (N1), $q = 2p-1$ in cases (N2) and (N3).

Remark 3.6 (Energies and norms of the solutions). Let $\mathscr{E}_n := \int_0^\pi (u_n)_t^2/2 + (u_n)_x^2/2 + F(u_n)\,\mathrm{d}x$ be the energy of u_n.

$$\|u_n\|_{H^1} = n\left(\frac{|\omega-1|n^2}{m_n(q+1)}\right)^{1/(q-1)}\left(1 + g_1\left(\frac{(|\omega-1|n^2)^{(p-1)/(q-1)}}{\gamma_\omega}\right)\right),$$

$$\mathscr{E}_n = \frac{n^2}{2\pi}\left(\frac{|\omega-1|n^2}{m_n(q+1)}\right)^{2/(q-1)}\left(1 + g_2\left(\frac{(|\omega-1|n^2)^{(p-1)/(q-1)}}{\gamma_\omega}\right)\right),$$

where $q = d$ if p is even and $d < 2p-1$ is the smallest order of the odd terms, $q = 2p-1$ if p is even and there is no odd term of order $< 2p-1$. Moreover, $\lim_{s\to 0} g_i(s) = 0$ and $m_n = m + O(1/n^2)$ if p is even, where $m > 0$ depends only on f. In the case when the smallest order of the odd terms is $2p-1$ and $0 < b < p\pi^2a^2/24$, we have two sequences of solutions which correspond to two (possibly) different values of m.

4. Regularity

We now prove that the weak solutions u_n actually are classical $C^2(\Omega)$ solutions of Eq. (1).

Define the norm

$$|u|_{\omega,3} := |u|_\infty + |\omega-1|^{1/2}\|u\|_{H^1} + |\omega-1|\,\|u\|_{H^2} + |\omega-1|^{3/2}\|u\|_{H^3}.$$

We saw in Lemma 2.2 that for $u\in X$, $|u|_\omega < \rho_0$, $|f(u)|_\infty \leqslant C|u|_\infty^p$ and $\|f(u)\|_{H^1} \leqslant C|u|_\infty^{p-1}\|u\|_{H^1}$. We have also the standard (although less obvious, stemming from the Gagliardo–Nirenberg inequalities) estimates $\|f(u)\|_{H^r} \leqslant C|u|_\infty^{p-1}\|u\|_{H^r}$ for $r=2,3$, from which we can derive

$$|f(u)|_{\omega,3} \leqslant C|u|_{\omega,3}^p \quad \text{for } u\in X, |u|_\omega < \rho_0. \tag{52}$$

Lemma 4.1. *If $u\in X\cap H^3(\Omega)$ then $w := L_\omega^{-1}\Pi_W u := \sum_{l\geqslant 0, j\geqslant 1, j\neq l} w_{lj}\cos(lt)\sin(jx)$, where $w_{lj} = u_{lj}/(\omega^2 l^2 - j^2)$, satisfies $\sum_{l\geqslant 0, j\geqslant 1, j\neq l}|w_{lj}|^2(l^6+j^6) \leqslant (C_{14}/\gamma)^2\|u\|_{H^3}^2$. In particular, $w\in H^3(\Omega)$, $\|w\|_{H^3} = O(\|u\|_{H^3}/\gamma)$ and $w_{xx}(t,0)=w_{xx}(t,\pi)=0$ a.e.. Moreover $w\in C^2(\bar\Omega)$.*

Proof. We point out that $u\in H^3(\Omega)$ does not imply that $\sum_{l\geqslant 0, j\geqslant 1}|u_{lj}|^2 j^6$ is finite (it is right only if also $u_{xx}(t,0)=u_{xx}(t,\pi)=0$ a.e.; in general we have only $\|u\|_{H^3}^2 \leqslant C\sum_{l\geqslant 0, j\geqslant 1}|u_{lj}|^2(l^6+j^6)$). However, it is true that $\sum_{l\geqslant 0, j\geqslant 1, j\neq l}|u_{lj}|^2 l^6 = O(|\partial_{ttt}u|_{L^2}^2) = O(\|u\|_{H^3}^2)$. Now

$$\sum_{l\geqslant 0, j\geqslant 1, j\neq l} w_{lj}^2(l^6+j^6) = \sum_{l\geqslant 0, j\geqslant 1, j\neq l}\frac{|u_{lj}|^2}{(\omega^2 l^2 - j^2)^2}(l^6+j^6) = S_1 + S_2, \tag{53}$$

where

$$S_1 := \sum_{j\geqslant 2\omega l}\frac{|u_{lj}|^2}{(\omega^2 l^2 - j^2)^2}(l^6+j^6) \quad \text{and} \quad S_2 := \sum_{j<2\omega l, j\neq l}\frac{|u_{lj}|^2}{(\omega^2 l^2 - j^2)^2}(l^6+j^6).$$

If $j \geqslant 2\omega l$ then $|\omega^2 l^2 - j^2| \geqslant 3j^2/4$, hence

$$S_1 \leqslant \sum_{j\geqslant 2\omega l}\frac{2u_{lj}^2}{j^4}\left(j^6 + \left(\frac{j}{2\omega}\right)^6\right) \leqslant \sum_{j\geqslant 2\omega l} 4|u_{lj}|^2 j^2 = O(|u_x|_{L^2}^2) = O(\|u\|_{H^1}^2). \tag{54}$$

Since $|\omega^2 l^2 - j^2| \geqslant \gamma/2$ for all $j\neq l$ and $\omega \leqslant \frac{3}{2}$ we have

$$\begin{aligned} S_2 &\leqslant \sum_{1\leqslant j<2\omega l}\frac{4|u_{lj}|^2}{\gamma^2}(l^6+j^6) \leqslant \sum_{1\leqslant j<2\omega l}\frac{4|u_{lj}|^2}{\gamma^2}(l^6+(3l)^6) \\ &\leqslant \frac{C}{\gamma^2}|u_{ttt}|_{L^2}^2 \leqslant \frac{C'}{\gamma^2}\|u\|_{H^3}^2. \end{aligned} \tag{55}$$

By (53)–(55), we get $\sum_{l\geqslant 0, j\geqslant 1, j\neq l}|w_{lj}|^2(l^6+j^6) \leqslant (C/\gamma)^2\|u\|_{H^3}^2$. Now we prove that

$$\sum_{l\geqslant 0, j\geqslant 1}|w_{lj}|(l^2+j^2) < +\infty, \tag{56}$$

which implies that $w \in C^2(\bar{\Omega})$. We have

$$\sum_{l \geqslant 0, j \geqslant 1, j \neq l} |w_{lj}|(l^2 + j^2) = \sum_{l \geqslant 0, j \geqslant 1, j \neq l} \frac{|u_{lj}|}{|\omega^2 l^2 - j^2|}(l^2 + j^2) = S_1' + S_2', \tag{57}$$

where

$$S_1' := \sum_{j \geqslant 2\omega l} \frac{|u_{lj}|}{|\omega^2 l^2 - j^2|}(l^2 + j^2) \quad \text{and} \quad S_2' := \sum_{j < 2\omega l, j \neq l} \frac{|u_{lj}|}{|\omega^2 l^2 - j^2|}(l^2 + j^2).$$

For $j \geqslant 2\omega l$, $|\omega^2 l^2 - j^2| \geqslant 3j^2/4$ and $l \leqslant j/2\omega \leqslant j$, hence

$$S_1' \leqslant C \sum_{j \geqslant 2\omega l} |u_{lj}| \leqslant C \left(\sum_{l \geqslant 0, j \geqslant 1} |u_{lj}|^2 (l^2 + j^2)^2 \right)^{1/2} \times \left(\sum_{l \geqslant 0, j \geqslant 1} \frac{1}{(l^2 + j^2)^2} \right)^{1/2} \leqslant C' \|u\|_{H^2}. \tag{58}$$

Note that $\sum_{l \geqslant 0, j \geqslant 1} |u_{lj}|^2 (l^2 + j^2)^2 = O(\|u\|^2_{H^2})$. We claim that

$$S_2' \leqslant \sum_{j < 2\omega l, j \neq l} C \frac{|u_{lj}| l^2}{|\omega^2 l^2 - j^2|} < +\infty. \tag{59}$$

Indeed we know that

$$\sum_{j < 2\omega l} (u_{lj} l^2)^2 l^2 = O(|\partial_{ttt} u|^2_{L^2}) = O(\|u\|^2_{H^3}) < +\infty. \tag{60}$$

As in the proof of (A.1) (proof of Lemma 2.1), (60) implies that S_2' is finite. By (57)–(59) we get (56). $\square$

By Lemma 4.1, $\|L_\omega^{-1} \Pi_W\|_{\omega,3} \leqslant C/\gamma$. Hence, by (52), the map $\mathscr{G}_v : W \to W$, defined (before Lemma 2.2) by $\mathscr{G}_v(w) := L_\omega^{-1} \Pi_W f(v + w)$, satisfies (if $v, w \in H^3(\Omega)$)

$$|\mathscr{G}_v(w)|_{\omega,3} \leqslant \frac{\tilde{C}}{\gamma}(|v|^p_{\omega,3} + |w|^p_{\omega,3}), \tag{61}$$

where $\tilde{C}$ is some positive constant.

Lemma 4.2. *If* $|v|_{\omega,3} \leqslant \tilde{\delta} := [\gamma/(2\tilde{C})]^{1/(p-1)}$ *then* $w(v) \in H^3(\Omega) \cap C^2(\bar{\Omega})$ *and so* $v + w(v) \in C^2(\bar{\Omega})$.

Proof. By (61), $\mathscr{G}_v(\tilde{B}_{\tilde{\delta}}) \subset \tilde{B}_{\tilde{\delta}}$, where

$$\tilde{B}_{\tilde{\delta}} := \{ w \in W \cap H^3(\Omega) \mid |w|_{\omega,3} \leqslant \tilde{\delta} \}.$$

As a result, if $|v|_{\omega,3} \leqslant \tilde{\delta}$ then $w_k := \mathscr{G}_v^k(0) \in \tilde{B}_{\tilde{\delta}}$ for all $k \geqslant 1$. It follows that (w_k) converges weakly in $H^3(\Omega)$ to some $\tilde{w} \in \tilde{B}_{\tilde{\delta}}$. Now, recall that $w(v)$ is the unique fixed point of $\mathscr{G}_v$ in a small ball around 0 for the norm $|\cdot|_\omega$ in W and that, by the contraction mapping principle, $w(v)$ is the H^1 limit of (w_k). Hence $w(v) = \tilde{w}$ and $w(v) \in H^3(\Omega)$. By formula (52) and Lemma 4.1 we derive that $w(v) \in C^2(\bar{\Omega})$, and $v + w(v) \in C^2(\bar{\Omega})$ because $V \cap H^3(\Omega) \subset C^2(\bar{\Omega})$. $\square$

There remains to check that when condition (35) or (36) or (45) or (51) is satisfied (according to the different cases) the critical points v_n of Φ_ω that we have obtained in Theorems 3.1–3.4 satisfy $|v_n|_{\omega,3} \leqslant \tilde{\delta}$. Since the critical points $v := v_n$ of Φ_ω satisfy $v_{tt} = v_{xx}$, $2\varepsilon v_{tt} = \Pi_V f(v + w(v))$,

$$\|v\|_{H^3} = O\left(\frac{1}{\varepsilon} \|\Pi_V f(v + w(v))\|\right) = O\left(\frac{1}{\varepsilon} \|f(v + w(v))\|\right),$$

$$\|v\|_{H^2} = O\left(\frac{1}{\varepsilon} |f(v + w(v))|_{L^2}\right).$$

Therefore, by Lemmas 2.2 and 2.1 (recall that $|\varepsilon| \sim |\omega - 1|$ as $\omega \to 1$)

$$|v|_{\omega,3} = O(|v|_\omega + |f(v + w(v))|_\omega) = O\left(|v|_\omega + |v|_\omega^p \left(1 + \frac{|v|_\omega^{p-1}}{\gamma}\right)\right) = O(|v|_\omega),$$

because $v \in \mathscr{D}_\rho := \{v \in V \mid |v|_\omega^{p-1}/\gamma < \rho\}$. Hence, by footnote 6 and Remark 3.5, $|v|_{\omega,3} \leqslant \tilde{\delta} := (\gamma/(2\tilde{C}))^{1/p-1}$ provided the constants C_7, C_8, C_9, C_{10} of Theorems 3.1, 3.2, 3.3 and 3.4 have been chosen small enough.

Acknowledgements

The authors thank A. Ambrosetti, D. Bambusi and M. Procesi for useful discussions. Part of this paper was written when the second author was visiting S.I.S.S.A. in Trieste.

Appendix A

Proof of Lemma 2.1. Writing $w(t,x) = \sum_{l \geqslant 0, j \geqslant 1, j \neq l} w_{l,j} \cos(lt)\sin(jx) \in W$ we have

$$L_\omega^{-1} w(t,x) = \sum_{l \geqslant 0, j \geqslant 1, j \neq l} \frac{w_{l,j}}{(\omega l - j)(\omega l + j)} \cos(lt)\sin(jx).$$

Since $\omega \in W_\gamma$, $|\omega^2 l^2 - j^2| = |(\omega l - j)(\omega l + j)| \geqslant \gamma/2$ and so $\|L_\omega^{-1} w\| = O(\|w\|/\gamma)$.

We now prove that for all $w \in W$

$$|L_\omega^{-1} w|_\infty \leqslant S := \sum_{l \geqslant 0, j \geqslant 1, j \neq l} \frac{|w_{l,j}|}{|\omega l - j|(\omega l + j)}$$

$$\leqslant C \left(|w|_{L^2} + \frac{|\omega - 1|^{1/2}}{\gamma} \|w\|_{H^1} \right). \tag{A.1}$$

For $l \in \mathbf{N}$, let $e(l) \in \mathbf{N}$ be defined by $|e(l) - \omega l| = \min_{j \in \mathbf{N}} |j - \omega l|$. Since ω is not rational, $e(l)$ is the only integer e such that $|e - \omega l| < \frac{1}{2}$. Let us define

$$S_1 := \sum_{l \geqslant 0, j \geqslant 1, j \neq l, j \neq e(l)} \frac{|w_{l,j}|}{|\omega l - j|(\omega l + j)} \quad \text{and}$$

$$S_2 := \sum_{l \geqslant 0, e(l) \neq l} \frac{|w_{l,e(l)}|}{|\omega l - e(l)|(\omega l + e(l))}.$$

For $j \neq e(l)$ we have

$$|\omega l - j| \geqslant \frac{|j - e(l)|}{2} \quad \text{and} \quad \omega l + j \geqslant \frac{|j - e(l)| + l}{4}. \tag{A.2}$$

Indeed, for $j \neq e(l)$ we have that $|j - \omega l| \geqslant |j - e(l)| - |e(l) - \omega l| \geqslant |j - e(l)| - \frac{1}{2} \geqslant |j - e(l)|/2$. Moreover, since $|e(l) - \omega l| < \frac{1}{2}$, it is easy to see (remember that $\omega \geqslant \frac{1}{2}$) that $e(l) + l \leqslant 4\omega l$ and hence $|j - e(l)| + l \leqslant j + e(l) + l \leqslant 4(j + \omega l)$. Defining $w_{l,j}$ by $w_{l,j} = 0$ if $j \leqslant 0$ or $j = l$, we get from (A.2)

$$S_1 \leqslant \sum_{l \geqslant 0, j \in \mathbf{Z}, j \neq e(l)} \frac{8|w_{l,j}|}{|j - e(l)|(|j - e(l)| + l)}.$$

Hence, by the Cauchy–Schwarz inequality, $S_1 \leqslant 8 R_1 |w|_{L^2}$, where

$$R_1^2 = \sum_{l \geqslant 0, j \in \mathbf{Z}, j \neq e(l)} \frac{1}{(j - e(l))^2 (|j - e(l)| + l)^2} = \sum_{l \geqslant 0, j \in \mathbf{Z}, j \neq 0} \frac{1}{j^2 (|j| + l)^2}$$

$$\leqslant \sum_{l \geqslant 0, j \in \mathbf{Z}, j \neq 0} \frac{1}{j^2 (1 + l)^2} < \infty.$$

In order to find an upper bound for S_2, we observe that if $l < 1/(2|\omega - 1|)$ then $|\omega l - l| < \frac{1}{2}$ hence $e(l) = l$. Hence we can write

$$S_2 = \sum_{l \geqslant 1/2|\omega - 1|, e(l) \neq l} \frac{|w_{l,e(l)}|}{|\omega l - e(l)|(\omega l + e(l))}.$$

Since $\omega \in W_\gamma$, for $l \neq e(l)$, $|\omega l - e(l)||\omega l + e(l)| \geqslant \gamma(\omega l + e(l)) l^{-1} \geqslant \gamma$. Hence, still by the Cauchy–Schwarz inequality,

$$S_2 \leqslant \frac{1}{\gamma} \sum_{l \geqslant 1/2|\omega - 1|, e(l) \neq l} |w_{l,e(l)}| \leqslant \frac{2R_2}{\gamma} \|w\|_{H^1},$$

where

$$R_2^2 = \sum_{l \geqslant 1/2|\omega-1|} \frac{1}{l^2} \leqslant 2|\omega - 1|.$$

The estimates $|L_\omega^{-1} w|_\omega \leqslant (C/\gamma)|w|_\omega \ \forall w \in W$ and $|L_\omega^{-1} \Pi_W u|_\omega \leqslant (C_1/\gamma)|u|_\omega \forall u \in X$ are an immediate consequence of the bound of $\|L_\omega^{-1} w\| = O(\|w\|/\gamma)$ and (A.1), since $|\Pi_W u|_{L_2} \leqslant |u|_{L_2}$ and $\|\Pi_W u\| \leqslant \|u\|$ For the estimate of $|L_\omega^{-1} w|_{L^2}$ ($w \in W$), we use the decomposition $|L_\omega^{-1} w|_{L^2}^2 = S_1' + S_2'$, where

$$S_1' := \sum_{j>0, l \geqslant 0, j \neq l, j \neq e(l)} \frac{w_{l,j}^2}{(\omega l - j)^2 (\omega l + j)^2},$$

$$S_2' := \sum_{l \geqslant 1/2|\omega-1|, e(l) \neq l} \frac{w_{l,e(l)}^2}{(\omega l - e(l))^2 (\omega l + e(l))^2}.$$

Recall that if $l < 1/2|\omega - 1|$ then $e(l) = l$. Arguing as before it is obvious that $S_1' \leqslant C|w|_{L^2}^2$. Setting $\varepsilon := (\omega^2 - 1)/2$, we have

$$S_2' \leqslant \sum_{l \geqslant 1/2|\omega-1|, e(l) \neq l} \frac{w_{l,e(l)}^2}{\gamma^2} \leqslant \frac{4|\omega - 1|^2}{\gamma^2} \sum_{l \geqslant 1/2|\omega-1|, e(l) \neq l} l^2 w_{l,e(l)}^2$$
$$= O\left(\frac{\varepsilon^2}{\gamma^2} \|w\|_{H^1}^2 \right) = O\left(\frac{|\varepsilon|}{\gamma^2} |w|_\omega^2 \right),$$

which yields estimate (7). We now prove (5). Writing $r(t,x) = \sum_{l \geqslant 0, j \geqslant 1} r_{l,j} \cos(lt) \sin(jx)$, $s(t,x) = \sum_{l \geqslant 0, j \geqslant 1} s_{l,j} \cos(lt) \sin(jx)$ we get

$$L_1^{-1} \Pi_W s = \sum_{j \neq l} \frac{s_{l,j}}{l^2 - j^2} \cos(lt) \sin(jx), \quad L_\omega^{-1} s = \sum_{j \neq l} \frac{s_{l,j}}{\omega^2 l^2 - j^2} \cos(lt) \sin(jx)$$

and

$$S := \int_\Omega r(L_\omega^{-1} - L^{-1})(\Pi_W s) \, \mathrm{d}t \, \mathrm{d}x = \pi^2 \sum_{j \neq l} \frac{s_{l,j} r_{l,j} \ (-\varepsilon) l^2}{(\omega^2 l^2 - j^2)(l^2 - j^2)} = -\pi^2 (S_1 + S_2),$$

where

$$S_1 := \sum_{j \neq l, j \neq e(l)} \frac{s_{l,j} r_{l,j} \ \varepsilon l^2}{(\omega l - j)(\omega l + j)(l - j)(l + j)},$$

$$S_2 := \sum_{l \neq e(l)} \frac{s_{l,e(l)} r_{l,e(l)} \ \varepsilon l^2}{(\omega l - e(l))(\omega l + e(l))(l - e(l))(l + e(l))}.$$

For $j \neq e(l)$ and $j \neq l$ we have that $|\omega l - j| \geqslant |j - e(l)|/2 \geqslant 1/2$ hence $|(\omega l - j)(\omega l + j)(l - j)(l + j)| \geqslant (1/2)\omega l^2$. We obtain

$$|S_1| \leqslant \frac{2|\varepsilon|}{\omega} \sum_{j \neq l} |r_{l,j}||s_{l,j}| = O\left(\frac{2|\varepsilon|}{\omega} |r|_{L^2}|s|_{L^2}\right) = O\left(\frac{2|\varepsilon|}{\omega} |r|_{\omega}|s|_{\omega}\right).$$

As before, for $\omega \in W_\gamma$ and $l \neq e(l)$, we have $|\omega l - e(l)| \geqslant \gamma/l$ and so $|(\omega l - e(l))(\omega l + e(l))(l - e(l))(l + e(l))| \geqslant \gamma l$. Moreover $l \neq e(l)$ implies $l \geqslant 1/(2|\omega - 1|)$. Hence

$$|S_2| \leqslant C \frac{\varepsilon^2}{\gamma} \sum_{l \geqslant 1/(2|\varepsilon|)} |r_{l,e(l)}||s_{l,e(l)}| l^2 = O\left(\frac{\varepsilon^2}{\gamma} \|r\|_{H^1}\|s\|_{H^1}\right).$$

Estimate (5) follows straightforwardly from the above estimates of S_1 and S_2. Finally, (6) is an immediate consequence of (5) since, for all $(r,s) \in X \times X$,

$$\int_\Omega r L^{-1}(\Pi_W s) = O(|r|_{L^2}|s|_{L^2}) = O(|r|_\omega |s|_\omega). \qquad \square$$

Proof of Lemma 2.2. By standard results, if $u \in L^\infty \cap H^1(\mathbf{T} \times (0,\pi))$, then $f(u) \in H^1(\mathbf{T} \times (0,\pi))$ and $(f(u))_x \equiv f'(u)u_x$, $(f(u))_t \equiv f'(u)u_t$. We get

$$\begin{aligned} |(f(u))_x|_{L^2} &= \left(\int_\Omega |(f(u))_x|^2\right)^{1/2} = \left(\int_\Omega |f'(u)|^2 |u_x|^2\right)^{1/2} \\ &\leqslant |f'(u)|_\infty \|u\|_{H^1} \leqslant C|u|_\infty^{p-1}\|u\|_{H^1} \end{aligned} \tag{A.3}$$

for $|u|_\infty$ small enough. An analogous estimate holds for $|(f(u))_t|_{L^2}$. Since $|f(u)| \leqslant C|u|_\infty^p$, we obtain (8).

We now prove that $u \to f(u)$ is $C^1(X,X)$. It is easy to show that it is Gateaux differentiable and that $D_G f(u)[h] = f'(u)h$. Moreover, with estimates similar as before we have that $u \to D_G f(u)$ is continuous. Hence, the Nemytski operator $u \to f(u)$ is in $C^1(X,X)$ and its Frechet differential is $h \to Df(u)[h] = D_G f(u)[h] = f'(u)h$. The second estimate of (8) can be obtained in the same way. $\square$

Proof of Lemma 2.3. We want to prove that $\mathscr{G} : W \to W$, defined by $\mathscr{G}(w) := L_\omega^{-1}\Pi_W f(v + w)$, is a contraction on a ball $B_\delta \subset W$ for some δ if $|v|_\omega^{p-1}/\gamma$ is small enough. Since, by Lemma 2.1, $\|L_\omega^{-1}\Pi_W\|_\omega \leqslant C_1/\gamma$ and by (8), we have that, for all $w \in B_\delta$

$$|\mathscr{G}(w)|_\omega \leqslant \frac{C_1}{\gamma} |f(v+w)|_\omega \leqslant \frac{C}{\gamma} |v+w|_\omega^p \leqslant \frac{C}{\gamma} (|v|_\omega + \delta)^p \leqslant \frac{\bar{C}}{\gamma} (|v|_\omega^p + \delta^p) \tag{A.4}$$

for some $\bar{C} > 0$. Moreover, by Lemma 2.2 the operator $\mathscr{G}$ is differentiable and $D\mathscr{G}(w)[h] = L_\omega^{-1}\Pi_W(f'(v+w)h)$. Hence by (A.1)

$$|D\mathscr{G}(w)[h]|_\omega \leqslant \frac{C_1}{\gamma} |f'(v+w)h|_\omega \leqslant \frac{C}{\gamma} |v+w|_\omega^{p-1}|h|_\omega \leqslant \frac{\tilde{C}}{\gamma} (|v|_\omega^{p-1} + \delta^{p-1})|h|_\omega$$

for some $\tilde{C} > 0$. Let us choose $\delta := |v|_\omega$ and $\rho := (4\max(\bar{C}, \tilde{C}))^{-1}$. For $v \in \mathscr{D}_\rho$, $(\bar{C}/\gamma)(|v|_\omega^p + \delta^p) \leqslant \delta/2$ and $(\tilde{C}/\gamma)(|v|_\omega^{p-1} + \delta^{p-1}) \leqslant \frac{1}{2}$. Hence, $\mathscr{G}$ maps B_δ into the closed ball $\bar{B}_{\delta/2}$ and is a contraction on B_δ. By the contraction mapping theorem there is a unique $w(v)$ in B_δ such that $\mathscr{G}(w(v)) = w(v)$. We have

$$|w(v)|_\omega = |\mathscr{G}(w(v))|_\omega \leqslant \frac{2\bar{C}}{\gamma} |v|_\omega^p \tag{A.5}$$

by (A.4) and the choice of δ. Moreover

$$\begin{aligned} |w(v)|_{L^2} = |L_\omega^{-1} \Pi_W f(v + w(v))|_{L^2} &\leqslant C \left(1 + \frac{\sqrt{|\omega - 1|}}{\gamma}\right) |f(v+w)|_\omega \\ &\leqslant C \left(1 + \frac{\sqrt{|\omega - 1|}}{\gamma}\right) |v|_\omega^p, \end{aligned}$$

by (7) and (A.5). In addition

$$\begin{aligned} \left| \int_\Omega r(w(v) - L_\omega^{-1} \Pi_W f(v)) \right| &= \left| \int_\Omega r L_\omega^{-1} \Pi_W (f(v + w(v)) - f(v)) \right| \\ &\leqslant C_2 \left(1 + \frac{|\omega - 1|}{\gamma}\right) |f(v + w(v)) - f(v)|_\omega |r|_\omega \\ &\leqslant C \left(1 + \frac{|\omega - 1|}{\gamma}\right) (|v|_\omega^{p-1} |w(v)|_\omega) |r|_\omega \\ &\leqslant \frac{C}{\gamma} \left(1 + \frac{|\omega - 1|}{\gamma}\right) |v|_\omega^{2p-1} |r|_\omega, \end{aligned}$$

by (6) and the bound on $|w(v)|_\omega$ given in (A.5). This proves (ii).

In order to prove (iii), let us call $\mathscr{I} : X \to X$ the linear operator defined by $\mathscr{I} u(t,x) = u(t + \pi, \pi - x)$. It is easy to see that $\mathscr{I}$ and $L_\omega^{-1} \Pi_W$ commute. Hence, since $-v = \mathscr{I} v$,

$$\begin{aligned} w(v) = L_\omega^{-1} \Pi_W f(v + w(v)) \Rightarrow \mathscr{I} w(v) &= L_\omega^{-1} \Pi_W \mathscr{I} f(v + w(v)) \\ &= L_\omega^{-1} \Pi_W f(-v + \mathscr{I} w(v)). \end{aligned}$$

This implies, by the uniqueness of the solution of $w = L_\omega^{-1} \Pi_W f(-v + w)$ in the ball of radius $|v|_\omega$, that $w(v)(t + \pi, \pi - x) = w(-v)(t,x)$.

For (iv), we remark that if $v \in V_n$, then $w(v)$ is the unique solution in an appropriate ball of the equation $\mathscr{G}(w) = w$, where the map $\mathscr{G}$ satisfies $\mathscr{G}(W_n) \subset W_n$ (W_n is the closed subspace of W containing $w \in W$ which are $2\pi/n$ periodic in time). Since $\mathscr{G}^k(0) \to w(v)$ in W, this implies that $w(v) \in W_n$.

Finally, the map $(v, w) \to \mathscr{G}(v; w) := w - L_\omega^{-1} \Pi_W f(v + w)$ is of class C^1 and its differential with respect to w is invertible at any point $(v, w(v))$ by Lemma 2.2 and the previous bounds. As a consequence of the implicit function theorem the map $v \to w(v)$ is in $C^1(\mathscr{D}_\rho, W)$. $\square$

Proof of Lemma 3.6. We define α, a_1, a_2, $\tilde{m}$ by $\alpha = \langle m \rangle$,

$$a_1(s_1) := \langle m(s_1, \cdot) - \alpha \rangle_{s_2} = \frac{1}{2\pi} \int_0^{2\pi} m(s_1, s_2) - \alpha \, \mathrm{d}s_2,$$

$$a_2(s_2) := \langle m(\cdot, s_2) - \alpha \rangle_{s_1} = \frac{1}{2\pi} \int_0^{2\pi} m(s_1, s_2) - \alpha \, \mathrm{d}s_1$$

and $\tilde{m}(s_1, s_2) := m(s_1, s_2) - a_1(s_1) - a_2(s_2) - \alpha$. It is straightforward to check that $\langle a_i \rangle = 0$, $\langle \tilde{m} \rangle_{s_2}(s_1) = 0$, $\langle \tilde{m} \rangle_{s_1}(s_2) = 0$. We have to prove that $a_1 = a_2$. Since $w \in W$, for all p odd and 2π periodic,

$$\int w(t,x)(p(t+x) - p(t-x)) \, \mathrm{d}t \, \mathrm{d}x = \frac{1}{2} \int_{\mathbf{T}^2} m(s_1, s_2)(p(s_1) - p(s_2)) \, \mathrm{d}s_1 \, \mathrm{d}s_2 = 0.$$

By the definition of a_i, we obtain that for all p odd and 2π-periodic

$$\int_{\mathbf{T}} (a_1(s) - a_2(s)) p(s) \, \mathrm{d}s = 0.$$

Therefore $a_1 - a_2$ is even. Now $w(t,x) = w(-t,x)$ for all $t \in \mathbf{T}$ and for all $x \in [0,\pi]$, and this implies that $m(-s_2, -s_1) = m(s_1, s_2)$. As a consequence $a_2(s) = a_1(-s)$, so that $a_1 - a_2$ is odd. Since $a_1 - a_2$ is both odd and even, $a_1 = a_2$.

We now turn to the expression of $\int wL^{-1}(w)$. Let us define β_1, β_2, β_3 on R^2 by

$$\beta_1(s_1, s_2) := -M(s_1, s_2) + \tfrac{1}{2}(M(s_1, s_1) + M(s_2, s_2)),$$

$$\beta_2(s_1, s_2) := (s_1 - s_2)(A(s_1) - A(s_2)),$$

$$\beta_3(s_1, s_2) := -\frac{\alpha}{8}(s_1 - s_2)(2\pi - (s_1 - s_2))$$

and B_1, B_2, B_3 on $R \times (0,\pi)$ by $B_i(t,x) := \beta_i(t+x, t-x)$. As an immediate consequence of its definition, B_i is 2π-periodic w.r.t. t and for $x \in \{0, \pi\}$, $B_i(t,x) = 0$. Hence $B_i \in X$. Moreover $-\partial_{s_1} \partial_{s_2} \beta_i = (1/4)\lambda_i$, where

$$\lambda_1(s_1, s_2) := \tilde{m}(s_1, s_2), \quad \lambda_2(s_1, s_2) := a(s_1) + a(s_2), \quad \lambda_3(s_1, s_2) := \alpha.$$

This implies that $[-(\partial_t)^2 + (\partial_x)^2] B_i = L_i$ where we have defined $L_i(t,x) := \lambda_i(t+x, t-x)$. As a result $B_1 + B_2 + B_3 \in L^{-1}w + V$ and

$$\int_{[0,2\pi]\times[0,\pi]} w(t,x) L^{-1}(w)(t,x) \, \mathrm{d}t \, \mathrm{d}x$$

$$= \int_{[0,2\pi]\times[0,\pi]} (B_1 + B_2 + B_3)(t,x)(L_1 + L_2 + L_3)(t,x) \, \mathrm{d}t \, \mathrm{d}x.$$

We observe that, since B_i vanishes for $x=0$ and $x=\pi$, for all i,j,

$$\int B_i L_j \,\mathrm{d}t\,\mathrm{d}x = \int B_i((B_j)_{tt} - (B_j)_{xx})\,\mathrm{d}t\,\mathrm{d}x$$

$$= \int ((B_i)_{tt} - (B_i)_{xx})B_j \,\mathrm{d}t\,\mathrm{d}x = \int L_i B_j \,\mathrm{d}t\,\mathrm{d}x.$$

Hence

$$\int wL^{-1}w = \int_{[0,2\pi]\times[0,\pi]} B_1L_1 + 2\int B_1L_2 + 2\int B_1L_3$$

$$+ \int B_2L_2 + 2\int B_2L_3 + \int B_3L_3.$$

We have

$$\int B_1L_1 = \frac{1}{2}\int_{\mathbf{T}^2}\left(-M(s_1,s_2) + \frac{1}{2}M(s_1,s_1) + \frac{1}{2}M(s_2,s_2)\right)\tilde{m}(s_1,s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2$$

$$= -\frac{1}{2}\int_{\mathbf{T}^2} M(s_1,s_2)\tilde{m}(s_1,s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2,$$

because $\langle \tilde{m}\rangle_{s_i} = 0$.

$$\int B_1L_2 = \frac{1}{2}\int_{\mathbf{T}^2}\left(-M(s_1,s_2) + \frac{1}{2}M(s_1,s_1) + \frac{1}{2}M(s_2,s_2)\right)(a(s_1)+a(s_2))\,\mathrm{d}s_1\,\mathrm{d}s_2$$

$$= \pi\int_{\mathbf{T}} M(s,s)a(s)\,\mathrm{d}s$$

because $\langle M\rangle_{s_i} = 0$ and $\langle a\rangle = 0$.

$$\int B_1L_3 = \frac{\alpha}{2}\int_{\mathbf{T}^2}\left(-M(s_1,s_2) + \frac{1}{2}M(s_1,s_1) + \frac{1}{2}M(s_2,s_2)\right) = -\pi\alpha\int_{\mathbf{T}} M(s,s)\,\mathrm{d}s$$

because $\langle M\rangle = 0$.

$$\int B_2L_2 = \int_0^{2\pi}\int_0^{\pi} 2x(A(t+x) - A(t-x))(a(t+x) + a(t-x))\,\mathrm{d}t\,\mathrm{d}x$$

$$= \int_0^{\pi}\mathrm{d}x\, 8x\int_0^{2\pi}(A(t+x) - A(t-x))(A'(t+x) + A'(t-x))\,\mathrm{d}t$$

$$= \int_0^{\pi}\mathrm{d}x\, 8x\int_0^{2\pi}\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{(A(t+x) - A(t-x))^2}{2} - A(t-x)^2\right]$$

$$+ 2A(t+x)A'(t-x)\,\mathrm{d}t$$

$$= \int_0^{\pi}\mathrm{d}x\, 16x\int_0^{2\pi} A(t+x)A'(t-x)\,\mathrm{d}t$$

$$= \int_0^{\pi} \mathrm{d}x\, 16x \int_0^{2\pi} A(s)A'(s-2x)\,\mathrm{d}s$$

$$= \int_0^{2\pi} \mathrm{d}s\, A(s) \int_0^{\pi} 16xA'(s-2x)\,\mathrm{d}x = \int_0^{2\pi} \mathrm{d}s A(s)[8xA(s-2x)]_{x=\pi}^{x=0}$$

$$= -8\pi \int_0^{2\pi} A(s)^2\,\mathrm{d}s.$$

In the fore-last line, we have integrated by parts (w.r.t. x) and used the fact that $\langle A\rangle=0$.

$$\int B_2L_3 = \alpha \int_0^{\pi} \mathrm{d}x 2x \int_0^{2\pi} (A(t+x)-A(t-x))\,\mathrm{d}t = 0,$$

still because $\langle A\rangle = 0$. At last

$$\int B_3L_3 = -\frac{\alpha}{8}\int_0^{2\pi}\int_0^{\pi} 2x(2\pi-2x)\,\mathrm{d}t\,\mathrm{d}x = -\frac{\alpha^2\pi^4}{6}$$

Summing up, we get (39). □

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